

ON BIRATIONAL GEOMETRY OF THE SPACE OF PARAMETRIZED RATIONAL CURVES IN GRASSMANNIANS

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ABSTRACT. In this paper, we study the birational geometry of the Quot schemes of trivial bundles on \mathbb{P}^1 by constructing small \mathbb{Q} -factorial modifications of the Quot schemes as suitable moduli spaces. We determine all the models which appear in the minimal model program on the Quot schemes. As a corollary, we show that the Quot schemes are Mori dream spaces and log Fano.

1. INTRODUCTION

Let V be an n -dimensional vector space over an algebraically closed field k . For fixed integers $d \geq 0$ and $0 \leq r \leq n - 1$, the moduli space $R^\circ := \text{Mor}_d(\mathbb{P}^1, \mathbb{G})$ parametrizes all morphisms $\mathbb{P}^1 \rightarrow \mathbb{G}$ of degree d , where \mathbb{G} is the Grassmannian of r -dimensional quotient spaces of V . Such morphism $\mathbb{P}^1 \rightarrow \mathbb{G}$ corresponds to a locally free quotient sheaf of $V_{\mathbb{P}^1} := V \otimes \mathcal{O}_{\mathbb{P}^1}$ of rank r and degree d . Thus we can compactify R° by the Quot scheme R which parametrizes all rank r , degree d quotient sheaves of $V_{\mathbb{P}^1}$.

In [St], Strømme proved many properties of R . For example, he showed that R is an irreducible rational smooth projective variety of dimension $nd + r(n - r)$. He also determined $\text{Pic}(R)$ and the nef cone $\text{Nef}(R) \subset N^1(R)_{\mathbb{R}} := (\text{Pic}(R)/\equiv) \otimes \mathbb{R}$ as follows.

If $r = n - 1$ or $d = 0$, R is $\mathbb{P}(V^\vee \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)$ or the Grassmannian \mathbb{G} respectively. Hence $\text{Pic}(R) \cong \mathbb{Z}$ and $\text{Nef}(R)$ is spanned by the ample generator.

If $0 \leq r \leq n - 2$ and $d \geq 1$, Strømme showed that there exist base point free line bundles α, β on R such that $\text{Pic}(R) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ and $\text{Nef}(R) = \mathbb{R}_{\geq 0}\alpha + \mathbb{R}_{\geq 0}\beta$.

On the other hand, Jow [Jo] determined the effective cone $\text{Eff}(R)$ of R by constructing two effective divisors which span $\text{Eff}(R)$. Venkatram [Ve] determined the movable cone $\text{Mov}(R)$ of R , and the stable base locus decomposition of $\text{Eff}(R)$.

Birational geometry of moduli spaces is studied in many papers, [ABCH], [Ch], [Ha], etc. The purpose of this paper is to investigate the birational geometry of R by constructing small \mathbb{Q} -factorial modifications of R as suitable moduli spaces. Recall the definition of small \mathbb{Q} -factorial modifications.

Definition 1.1 ([HK, 1.8 Definition]). By a *small \mathbb{Q} -factorial modification (SQM)* of a projective variety X , we mean a birational map $f : X \dashrightarrow X'$ with X' projective, normal, and \mathbb{Q} -factorial, such that f is an isomorphism in codimension one. We note that such f induces an isomorphism $f^* : N^1(X')_{\mathbb{R}} \rightarrow N^1(X)_{\mathbb{R}}$ by pullbacks of divisors if X is \mathbb{Q} -factorial.

2010 *Mathematics Subject Classification.* 14C20, 14M99.

Key words and phrases. Quot scheme, small \mathbb{Q} -factorial modification, Mori dream space.

We can find SQMs of R as moduli spaces parametrizing following objects.

For a scheme T , we denote the second projection $\mathbb{P}^1 \times T \rightarrow T$ by π_T , or simply π . For a coherent sheaf \mathcal{F} on $\mathbb{P}^1 \times T$ and $m \in \mathbb{Z}$, we denote $\mathcal{F} \otimes p_1^* \mathcal{O}_{\mathbb{P}^1}(m)$ by $\mathcal{F}(m)$, where $p_1 : \mathbb{P}^1 \times T \rightarrow \mathbb{P}^1$ is the first projection.

Definition 1.2. Fix integers $0 \leq r \leq n-1$, $d \geq 0$, and $m \geq \lceil d/s \rceil$ for $s := n-r$. For a locally noetherian k -scheme T , a morphism $\iota : \mathcal{E} \rightarrow V_{\mathbb{P}^1 \times T} := V \otimes_k \mathcal{O}_{\mathbb{P}^1 \times T}$ of sheaves on $\mathbb{P}^1 \times T$ satisfies condition (\star_m) if it satisfies the following three conditions:

- i) \mathcal{E} is locally free of rank s and the degree of $\mathcal{E}|_{\mathbb{P}^1 \times \{t\}}$ is $-d$ for any $t \in T$,
- ii) $R^1 \pi_*(\mathcal{E}(m-1)) = 0$, or equivalently, $H^1(\mathcal{E}(m-1)|_{\mathbb{P}^1 \times \{t\}}) = 0$ for any $t \in T$,
- iii) the induced map $\pi_*(\mathcal{E}(m)) \otimes k(t) \rightarrow \pi_*(V_{\mathbb{P}^1 \times T}(m)) \otimes k(t) = V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(m))$ is injective for any $t \in T$.

For $\iota : \mathcal{E} \rightarrow V_{\mathbb{P}^1 \times T}$ and $\iota' : \mathcal{E}' \rightarrow V_{\mathbb{P}^1 \times T}$ which satisfy (\star_m) , we say that $\iota : \mathcal{E} \rightarrow V_{\mathbb{P}^1 \times T}$ and $\iota' : \mathcal{E}' \rightarrow V_{\mathbb{P}^1 \times T}$ are equivalent if and only if there exists an isomorphism $\mathcal{E} \rightarrow \mathcal{E}'$ such that

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\iota} & V_{\mathbb{P}^1 \times T} \\ \downarrow & & \parallel \\ \mathcal{E}' & \xrightarrow{\iota'} & V_{\mathbb{P}^1 \times T} \end{array}$$

is commutative. We denote by $[\mathcal{E} \rightarrow V_{\mathbb{P}^1 \times T}]$ the equivalence class of $\mathcal{E} \rightarrow V_{\mathbb{P}^1 \times T}$.

Theorem 1.3. Let n, r, d , and s be as in Definition 1.2. For each $m \geq \lceil d/s \rceil$, there exists a smooth projective variety R_m which is the fine moduli space parametrizing equivalence classes $[\mathcal{E} \rightarrow V_{\mathbb{P}^1 \times T}]$ which satisfy (\star_m) for each locally noetherian k -scheme T . Furthermore, there exists a natural birational map $\tilde{g}_m : R \dashrightarrow R_m$.

Remark 1.4. For each $m \geq d$, Strømme [St] defined a subscheme R_m (he denoted it by Z simply) in the direct product of two Grassmannians, and constructed an isomorphism $\tilde{g}_m : R \rightarrow R_m$ (in particular, all R_m are isomorphic for $m \geq d$). We use the same construction to define R_m and $\tilde{g}_m : R \dashrightarrow R_m$ in Theorem 1.3 for $m \geq \lceil d/s \rceil$. In this sense, the most essential part of this paper is already written in [St].

As stated in Remark 1.4, R_m is constructed as a subscheme of the direct product of two Grassmannians. Hence we have two projections from R_m to the Grassmannians. By investigating the projections, we can determine the nef cone of R_m as in the following theorems. In particular, this gives an alternative proof of the descriptions of $\text{Eff}(R)$, $\text{Mov}(R)$, and the stable base locus decomposition of $\text{Eff}(R)$ in [Jo], [Ve]. In the following theorems, α, β are the base point free line bundles on R defined by Strømme (see Section 5 for the definitions of α, β).

First, we describe the cases $r = 0$ or 1 .

Theorem 1.5. Assume $r \leq n-2$ and $d \geq 1$, and set $s = n-r$. If $r = 0$ or 1 , the following hold.

- (1) For each $\lfloor d/s \rfloor + 1 \leq m \leq d-1$, $\tilde{g}_m : R \dashrightarrow R_m$ is an SQM of R and

$$\text{Nef}(R_m) = \mathbb{R}_{\geq 0}(-(d-m-1)\alpha + \beta) + \mathbb{R}_{\geq 0}(-(d-m)\alpha + \beta)$$

holds, where we regard $\text{Nef}(R_m)$ as a subset of $N^1(R)_{\mathbb{R}}$ by $\tilde{g}_m^* : N^1(R_m)_{\mathbb{R}} \xrightarrow{\sim} N^1(R)_{\mathbb{R}}$. Both $-(d-m-1)\alpha + \beta$ and $-(d-m)\alpha + \beta$ are base point free on R_m .

(2) *It holds that*

$$\begin{aligned} \text{Mov}(R) &= \text{Nef}(R) \cup \bigcup_{m=\lfloor d/s \rfloor + 1}^{d-1} \text{Nef}(R_m) \\ &= \mathbb{R}_{\geq 0}\alpha + \mathbb{R}_{\geq 0}(-(d - \lfloor d/s \rfloor - 1)\alpha + \beta), \\ \text{Eff}(R) &= \mathbb{R}_{\geq 0}(2d\alpha - r\beta) + \mathbb{R}_{\geq 0}(-(d - \lceil d/s \rceil)\alpha + \beta) \end{aligned}$$

as in Figure 1.

(3) *The base point free line bundle α on R defines a surjective morphism*

$$f : R \rightarrow \mathbb{P}\left(\bigwedge^r V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee\right),$$

which is a fiber type contraction (resp. a divisorial contraction) for $r = 0$ (resp. $r = 1$).

(4) *The base point free class $-(d - \lfloor d/s \rfloor - 1)\alpha + \beta$ on $R_{\lfloor d/s \rfloor + 1}$ defines a surjective morphism to a Grassmannian*

$$R_{\lfloor d/s \rfloor + 1} \rightarrow \text{Gr}\left((\lfloor d/s \rfloor + 1)s - d, V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(\lfloor d/s \rfloor))\right),$$

which is a fiber type contraction (resp. a divisorial contraction) if $d/s \notin \mathbb{N}$ (resp. $d/s \in \mathbb{N}$).

(5) *For each $\lfloor d/s \rfloor + 1 \leq m \leq d - 1$, the class $-(d - m - 1)\alpha + \beta$, which is base point free on R_m and R_{m+1} (for $m = d - 1$, we identify $R_{m+1} = R_d$ with R by the isomorphism \tilde{g}_d), defines birational morphisms pr_1, pr_2 as in the diagram*

$$\begin{array}{ccc} R_m & \dashrightarrow & R_{m+1} \\ & \searrow \text{pr}_2 & \swarrow \text{pr}_1 \\ & X_m^0 & \end{array}$$

where X_m^0 is a degeneracy locus in a Grassmannian (see Definition 4.3 for the definition of X_m^0). Both pr_1 and pr_2 in the diagram are small contractions.

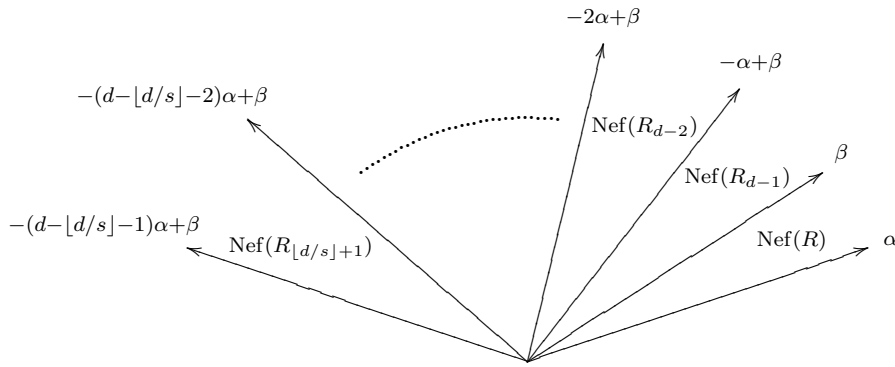


FIGURE 1. Decomposition of $\text{Mov}(R)$ for $r = 0, 1$

Next, we consider the cases $2 \leq r \leq n-2$. Let R' be the Quot scheme parametrizing all rank $n-r$, degree d quotient sheaves of the dual bundle $V_{\mathbb{P}^1}^\vee$. As we will see in Section 5, there exists a natural birational map $R \dashrightarrow R'$. Applying Theorem 1.3 to R' , we obtain a moduli space $R'_{m'}$ and a birational map $\tilde{g}'_m : R' \dashrightarrow R'_{m'}$ for each $m' \geq \lceil d/r \rceil$.

Theorem 1.6. *Assume $2 \leq r \leq n-2$ and $d \geq 1$, and set $s = n-r$. Then*

- (1) *For $\lfloor d/s \rfloor + 1 \leq m \leq d-1$ and $\lfloor d/r \rfloor + 1 \leq m' \leq d-1$, R' , R_m , and $R'_{m'}$ are SQMs of R , and it holds that*

$$\begin{aligned} \text{Nef}(R_m) &= \mathbb{R}_{\geq 0}(-(d-m-1)\alpha + \beta) + \mathbb{R}_{\geq 0}(-(d-m)\alpha + \beta), \\ \text{Nef}(R') &= \mathbb{R}_{\geq 0}(2d\alpha - \beta) + \mathbb{R}_{\geq 0}\alpha, \\ \text{Nef}(R'_{m'}) &= \mathbb{R}_{\geq 0}((d+m'+1)\alpha - \beta) + \mathbb{R}_{\geq 0}((d+m')\alpha - \beta). \end{aligned}$$

The classes $-(d-m-1)\alpha + \beta, -(d-m)\alpha + \beta$ are base point free on R_m , $2d\alpha - \beta, \alpha$ are base point free on R' , and $(d+m'+1)\alpha - \beta, (d+m')\alpha - \beta$ are base point free on $R'_{m'}$.

- (2) *It holds that*

$$\begin{aligned} \text{Mov}(X) &= \text{Nef}(R) \cup \text{Nef}(R') \cup \bigcup_{m=\lfloor d/s \rfloor + 1}^{d-1} \text{Nef}(R_m) \cup \bigcup_{m'=\lfloor d/r \rfloor + 1}^{d-1} \text{Nef}(R'_{m'}) \\ &= \mathbb{R}_{\geq 0}((d + \lfloor d/r \rfloor + 1)\alpha - \beta) + \mathbb{R}_{\geq 0}(-(d - \lfloor d/s \rfloor - 1)\alpha + \beta), \\ \text{Eff}(X) &= \mathbb{R}_{\geq 0}((d + \lceil d/r \rceil)\alpha - \beta) + \mathbb{R}_{\geq 0}(-(d - \lceil d/s \rceil)\alpha + \beta) \end{aligned}$$

as in Figure 2.

- (3) *The same statement as (4) in Theorem 1.5 holds.*
 (4) *The base point free class $(d + \lfloor d/r \rfloor + 1)\alpha - \beta$ on $R'_{\lfloor d/r \rfloor + 1}$ defines a surjective morphism to a Grassmannian*

$$R'_{\lfloor d/r \rfloor + 1} \twoheadrightarrow \text{Gr}\left(\left(\lfloor d/r \rfloor + 1\right)r - d, V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(\lfloor d/r \rfloor))\right),$$

which is a fiber type contraction (resp. a divisorial contraction) if $d/r \notin \mathbb{N}$ (resp. $d/r \in \mathbb{N}$).

- (5) *The same statement as (5) in Theorem 1.5 holds.*
 (6) *For each $\lfloor d/r \rfloor + 1 \leq m' \leq d-1$, the class $(d+m'+1)\alpha - \beta$, which is base point free on $R'_{m'}$ and $R'_{m'+1}$ (for $m' = d-1$, we identify $R'_{m'+1} = R'_d$ with R'), defines birational morphisms pr_1, pr_2 as in the diagram*

$$\begin{array}{ccc} R'_m & \dashrightarrow & R'_{m'+1} \\ & \searrow \text{pr}'_2 & \swarrow \text{pr}'_1 \\ & X_m'^0 & \end{array}$$

where $X_m'^0$ is a degeneracy locus in a Grassmannian. Both pr'_1 and pr'_2 in the diagram are small contractions.

- (7) The class α , which is base point free on R and R' , defines birational morphisms f, f' as in the diagram

$$\begin{array}{ccc} R & \dashrightarrow & R' \\ & \searrow f & \swarrow f' \\ & K_{s,r}^d & \end{array}$$

where $K_{s,r}^d$, which is defined to be the image of f , is called a quantum Grassmannian. Both f and f' in the diagram are small contractions.

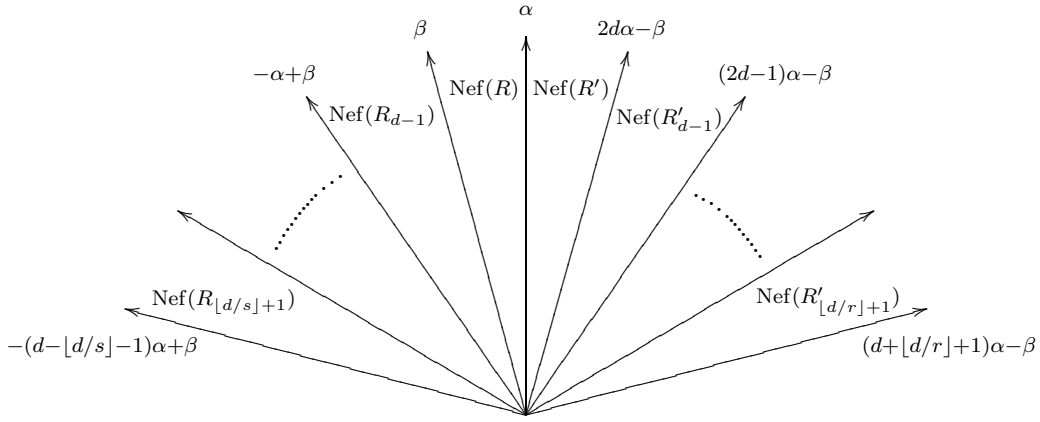


FIGURE 2. Decomposition of $\text{Mov}(R)$ for $2 \leq r \leq n - 2$

Mori dream spaces, which were introduced by Hu and Keel in [HK], are special varieties which have very nice properties in view of the minimal model program. A normal projective variety X is called a Mori dream space if

- 1) X is \mathbb{Q} -factorial and $\text{Pic}(X) \otimes \mathbb{Q} = N^1(X)_{\mathbb{Q}}$,
- 2) $\text{Nef}(X)$ is spanned by finitely many semiample line bundles,
- 3) there exists a finite corrections of SQMs $f_i : X \dashrightarrow X_i$ such that each X_i satisfies 2) and $\text{Mov}(X) = \bigcup_i f_i^*(\text{Nef}(X_i))$.

By Theorems 1.5, 1.6, we obtain the following corollary.

Corollary 1.7. *For $0 \leq r \leq n - 1$ and $d \geq 0$, the Quot scheme R is a Mori dream space. Moreover, R is log Fano, i.e., there exists an effective \mathbb{Q} -divisor D on R such that (R, D) is Kawamata log terminal and $-(K_R + D)$ is ample.*

Remark 1.8. If the characteristic of the base field k is zero, any \mathbb{Q} -factorial log Fano variety is a Mori dream space by [BCHM, Corollary 1.3.2].

This paper is organized as follows. In Section 2, we recall the definition of R_m in [St] and show that R_m parametrizes $[\mathcal{E} \rightarrow V_{\mathbb{P}^1}]$ satisfying (\star_m) . In Section 3, we prove the rest part of Theorem 1.3. In Section 4, we investigate morphisms from R_m to Grassmannians. In Section 5, we give the proof of Theorems 1.5, 1.6. Throughout this paper, we work over a fixed algebraically closed field k of any characteristic.

Acknowledgments. The author would like to express his gratitude to Professors Yoshinori Gongyo, Yasunari Nagai, and Yujiro Kawamata for valuable comments and advice. The author is supported by JSPS KAKENHI Grant Number 261881.

2. CONSTRUCTION OF R_m

Let V be an n -dimensional k -vector space. We denote $V \otimes \mathcal{O}_T$ by V_T for a k -scheme T . For fixed integers $0 \leq r \leq n-1$ and $d \geq 0$, let R be the Quot scheme parametrizing all rank r , degree d quotient sheaves of $V_{\mathbb{P}^1}$ as in Introduction. Set $s = n - r$. We denote by $\text{Gr}(s, V)$ the Grassmannian of s -dimensional subspaces of V .

For $m \geq 0$, we set $V_m = V \otimes_k H^0(\mathcal{O}_{\mathbb{P}^1}(m))$. Let $G_m = \text{Gr}((m+1)s - d, V_m)$ for $m \in \mathbb{Z}$ such that $(m+1)s - d \geq 0$, that is, for $m \geq \lceil d/s \rceil - 1$.

As stated in Introduction, Strømme constructed a subscheme $R_m \subset G_{m-1} \times G_m$ and an isomorphism $R \rightarrow R_m$ for each $m \geq d$ in [St, Section 4]. By the same construction, we can define R_m for $m \geq \lceil d/s \rceil$. Hence, recall his definition.

Let

$$(2.1) \quad 0 \rightarrow \mathcal{A} \rightarrow V_{\mathbb{P}^1 \times R} \rightarrow \mathcal{B} \rightarrow 0$$

be the universal exact sequence on $\mathbb{P}^1 \times R$. By tensoring $\mathcal{O}_{\mathbb{P}^1}(m)$ on (2.1) and pushing forward by π , we obtain a (not necessarily exact) sequence on R

$$(*_m) \quad 0 \rightarrow \mathcal{A}_m \rightarrow (V_m)_R \rightarrow \mathcal{B}_m \rightarrow 0$$

for each m , where $\mathcal{A}_m = \pi_*(\mathcal{A}(m))$, $\mathcal{B}_m = \pi_*(\mathcal{B}(m))$. We note that \mathcal{B}_m is locally free of rank $(m+1)r + d$ for $m \geq -1$. By the universal property of Grassmannians, we have rational maps as follows.

Definition 2.1 ([St, Section 4]). For $m \geq \lceil d/s \rceil - 1$, we denote by g_m the rational map $R \dashrightarrow G_m$ induced by the sequence $(*_m)$.

Remark 2.2. If $H^1(\mathcal{A}(m)|_{\mathbb{P}^1 \times \{z\}}) = 0$ for a point $z \in R$, $(*_m)$ is exact at z . Hence the rational map g_m is defined at z and $g_m(z)$ is the point in $G_m = \text{Gr}((m+1)s - d, V_m)$ corresponding to the subspace $H^0(\mathcal{A}(m)|_{\mathbb{P}^1 \times \{z\}}) \subset V_m$. Hence g_m is a morphism for $m \geq d-1$ as stated in [St, Section 4]. Moreover, g_m is defined on a nonempty open subset of R for each $m \geq \lceil d/s \rceil - 1$. To see this, we take an injection

$$\iota : \mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(-\lceil d/s \rceil)^{\oplus s-l} \oplus \mathcal{O}_{\mathbb{P}^1}(-\lceil d/s \rceil + 1)^{\oplus l} \rightarrow V_{\mathbb{P}^1}$$

for $l = \lceil d/s \rceil s - d \geq 0$. Then g_m is defined around $z = [V_{\mathbb{P}^1} \rightarrow \text{coker } \iota] \in R$ since $\mathcal{A}|_{\mathbb{P}^1 \times \{z\}} \cong \mathcal{E}$ and $H^1(\mathcal{E}(m)) = 0$ for $m \geq \lceil d/s \rceil - 1$.

Theorem 2.3 ([St, Theorem 4.2]). *For $m \geq d$, the morphism $g_m : R \rightarrow G_m$ is a closed embedding.*

Checking the proof of Theorem 2.3, we can obtain the following lemma.

Lemma 2.4. *For $m \geq \lceil d/s \rceil$, the rational map $g_m : R \dashrightarrow g_m(R) \subset G_m$ is birational.*

Proof. By the argument in the proof of Theorem 4.2 in [St], the birationality of g_m follows if $\mathcal{A}|_{\mathbb{P}^1 \times \{z\}}(m)$ is globally generated for general $z \in R$. Since $\mathcal{A}|_{\mathbb{P}^1 \times \{z\}}(m)$ is globally generated if and only if $H^1(\mathcal{A}|_{\mathbb{P}^1 \times \{z\}}(m-1)) = 0$, it is enough to show that $\mathcal{A}|_{\mathbb{P}^1 \times \{z\}}(m)$ is globally generated for *some* $z \in R$ by the upper semicontinuity

of cohomology. For $z = [V_{\mathbb{P}^1} \rightarrow \text{coker } \iota] \in R$ in Remark 2.2, $\mathcal{A}_{|\mathbb{P}^1 \times \{z\}}(m) \cong \mathcal{E}(m)$ is globally generated for $m \geq \lceil d/s \rceil$. \square

Remark 2.5. Contrary to the case $m \geq \lceil d/s \rceil$, the rational map $g_{\lceil d/s \rceil - 1}$ is not birational for $(r, d) \neq (0, 0)$ as follows.

Let $l = \lceil d/s \rceil s - d$. Since $G_{\lceil d/s \rceil - 1} = \text{Gr}(l, V_{\lceil d/s \rceil - 1})$, it holds that $\dim G_{\lceil d/s \rceil - 1} = l(n \lceil d/s \rceil - l)$. By $\dim R = nd + rs$, $n = r + s$, and $l = \lceil d/s \rceil s - d$, we can check that

$$\dim R_m - \dim G_{\lceil d/s \rceil - 1} = (s - l)((\lceil d/s \rceil + 1)r + d),$$

which is positive since $s - l > 0$ and $(r, d) \neq (0, 0)$. See also Proposition 4.8.

The following proposition is useful to study R .

Proposition 2.6 ([St, Proposition 1.1]). *Let T be a locally noetherian k -scheme and let \mathcal{F} be a coherent sheaf on $\mathbb{P}^1 \times T$, flat over T . Assume $R^1\pi_*(\mathcal{F}(-1)) = 0$. Then there is a short exact sequence on $\mathbb{P}^1 \times T$*

$$0 \rightarrow \pi^*(\pi_*\mathcal{F}(-1))(-1) \rightarrow \pi^*\pi_*\mathcal{F} \rightarrow \mathcal{F} \rightarrow 0.$$

This sequence is functorial in \mathcal{F} , and its formulation commutes with arbitrary base change $T' \rightarrow T$. Its first two terms are locally free.

By Proposition 2.6, there is an exact sequence

$$0 \rightarrow V_{m-1} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow V_m \otimes \mathcal{O}_{\mathbb{P}^1} \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^1}(m) \rightarrow 0$$

on \mathbb{P}^1 for each $m \geq 0$, which induces an injection $j_m : V_{m-1} \rightarrow V_m \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))$.

We denote the universal sequence on G_m for $m \geq \lceil d/s \rceil - 1$ by

$$(**_m) \quad 0 \rightarrow \mathcal{K}_m \rightarrow (V_m)_{G_m} \rightarrow \mathcal{Q}_m \rightarrow 0.$$

Fix $m \geq \lceil d/s \rceil$. Pulling back the sequences $(**_{m-1})$ and $(**_m)$ to $G_{m-1} \times G_m$, and tensoring the latter by $H := H^0(\mathcal{O}_{\mathbb{P}^1}(1))$, we obtain a diagram of locally free sheaves on $G_{m-1} \times G_m$:

$$(2.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{pr}_1^* \mathcal{K}_{m-1} & \xrightarrow{i_{m-1}} & (V_{m-1})_{G_{m-1} \times G_m} & \longrightarrow & \text{pr}_1^* \mathcal{Q}_{m-1} \longrightarrow 0 \\ & & & & \downarrow j_m & & \\ 0 & \longrightarrow & \text{pr}_2^* \mathcal{K}_m \otimes H & \longrightarrow & (V_m)_{G_{m-1} \times G_m} \otimes H & \xrightarrow{p_m} & \text{pr}_2^* \mathcal{Q}_m \otimes H \longrightarrow 0, \end{array}$$

where pr_1, pr_2 are the projections from $G_{m-1} \times G_m$ to G_{m-1}, G_m respectively. Now we can define R_m as follows.

Definition 2.7 ([St, Section 4]). For $m \geq \lceil d/s \rceil$, we denote by $R_m \subset G_{m-1} \times G_m$ the closed subscheme defined by the vanishing of $p_m \circ j_m \circ i_{m-1}$.

By definition, there is the following commutative diagram on R_m

$$(2.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & (\mathcal{K}_{m-1})_{R_m} & \longrightarrow & (V_{m-1})_{R_m} & \longrightarrow & (\mathcal{Q}_{m-1})_{R_m} \longrightarrow 0 \\ & & \downarrow & & \downarrow j_m & & \downarrow \\ 0 & \longrightarrow & (\mathcal{K}_m)_{R_m} \otimes H & \longrightarrow & (V_m)_{R_m} \otimes H & \longrightarrow & (\mathcal{Q}_m)_{R_m} \otimes H \longrightarrow 0, \end{array}$$

where $(\mathcal{K}_{m-1})_{R_m}, (\mathcal{K}_m)_{R_m}, (\mathcal{Q}_{m-1})_{R_m}, (\mathcal{Q}_m)_{R_m}$ are the restrictions of $\text{pr}_1^* \mathcal{K}_{m-1}, \text{pr}_2^* \mathcal{K}_m, \text{pr}_1^* \mathcal{Q}_{m-1}, \text{pr}_2^* \mathcal{Q}_m$ on R_m respectively. Pulling back this diagram to $\mathbb{P}^1 \times$

R_m by $\pi = \pi_{R_m}$, tensoring the lower sequence with the natural morphism $H \otimes \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\mathbb{P}^1}(1)$, and tensoring the whole diagram by $\mathcal{O}_{\mathbb{P}^1}(-1)$, we obtain the following diagram on $\mathbb{P}^1 \times R_m$

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \pi^*(\mathcal{K}_{m-1})_{R_m}(-1) & \longrightarrow & (V_{m-1})_{\mathbb{P}^1 \times R_m}(-1) & \longrightarrow & \pi^*(\mathcal{Q}_{m-1})_{R_m}(-1) \longrightarrow 0 \\
 & & \downarrow a & & \downarrow b & & \downarrow \\
 (2.4) \quad 0 & \longrightarrow & \pi^*(\mathcal{K}_m)_{R_m} & \longrightarrow & (V_m)_{\mathbb{P}^1 \times R_m} & \longrightarrow & \pi^*(\mathcal{Q}_m)_{R_m} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathcal{K}(m) & \xrightarrow{c} & V_{\mathbb{P}^1 \times R_m}(m) & \longrightarrow & \mathcal{Q}(m) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0,
 \end{array}$$

where \mathcal{K} and \mathcal{Q} are defined by the cokernels of the vertical maps.

By tensoring c in the diagram (2.4) with $\mathcal{O}_{\mathbb{P}^1}(-m)$, we have a morphism $\mathcal{K} \rightarrow V_{\mathbb{P}^1 \times R_m}$. In the proof of Theorem 4.1 in [St], Strømme showed that $\mathcal{K} \rightarrow V_{\mathbb{P}^1 \times R_m}$ is injective after restricting to each fiber of $\pi : \mathbb{P}^1 \times R_m \rightarrow R_m$ if $m \geq d$. For $m \geq \lceil d/s \rceil$, we have the following lemma.

Lemma 2.8. *For $m \geq \lceil d/s \rceil$, the morphism $\mathcal{K} \rightarrow V_{\mathbb{P}^1 \times R_m}$ satisfies condition (\star_m) .*

Proof. First note that a in the diagram (2.4) is pointwise injective since so is b . Hence \mathcal{K} is locally free on R_m . For $z \in R_m$, we have an exact sequence

$$0 \rightarrow ((\mathcal{K}_{m-1})_{R_m} \otimes k(z)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow ((\mathcal{K}_m)_{R_m} \otimes k(z)) \otimes \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{K}(m)|_{\mathbb{P}^1 \times \{z\}} \rightarrow 0$$

by restricting the left column in the diagram (2.4) on $\mathbb{P}^1 \times \{z\}$. By this exact sequence, $H^1(\mathcal{K}(m-1)|_{\mathbb{P}^1 \times \{z\}}) = 0$ holds. Hence ii) in (\star_m) is satisfied. Since

$$\dim(\mathcal{K}_{m-1})_{R_m} \otimes k(z) = ms - d, \quad \dim(\mathcal{K}_m)_{R_m} \otimes k(z) = (m+1)s - d,$$

$\mathcal{K}|_{\mathbb{P}^1 \times \{z\}}$ is of rank s and degree $-d$. Thus $\mathcal{K} \rightarrow V_{\mathbb{P}^1 \times R_m}$ satisfies i) in (\star_m) . Furthermore, the linear map

$$H^0(\mathcal{K}(m)|_{\mathbb{P}^1 \times \{z\}}) \cong \mathcal{K}_m \otimes k(z) \rightarrow V_m = V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(m))$$

induced by c is injective. Thus iii) in (\star_m) holds. \square

Using the above diagrams, Strømme proved the following theorem.

Theorem 2.9 ([St, Theorem 4.1]). *The morphism*

$$(g_{m-1}, g_m) : R \rightarrow G_{m-1} \times G_m$$

is an isomorphism onto R_m for $m \geq d$.

By a similar argument to the proof of Theorem 2.9, we can show that R_m are moduli spaces as in Theorem 1.3 for $m \geq \lceil d/s \rceil$ as follows.

Proposition 2.10. *For $m \geq \lceil d/s \rceil$, R_m is the fine moduli space parametrizing equivalence classes $[\mathcal{E} \rightarrow V_{\mathbb{P}^1 \times T}]$ which satisfy (\star_m) for each locally noetherian k -scheme T .*

Proof. For any morphism $\varphi : T \rightarrow R_m$, we obtain $(\text{id}_{\mathbb{P}^1} \times \varphi)^* \mathcal{K} \rightarrow V_{\mathbb{P}^1 \times T}$ by pulling back $\mathcal{K} \rightarrow V_{\mathbb{P}^1 \times R_m}$ in (2.4). By Lemma 2.8, $(\text{id}_{\mathbb{P}^1} \times \varphi)^* \mathcal{K} \rightarrow V_{\mathbb{P}^1 \times T}$ satisfies (\star_m) and we obtain an equivalence class $[(\text{id}_{\mathbb{P}^1} \times \varphi)^* \mathcal{K} \rightarrow V_{\mathbb{P}^1 \times T}]$.

Conversely, we can construct a morphism $T \rightarrow R_m$ from $[\iota : \mathcal{E} \rightarrow V_{\mathbb{P}^1 \times T}]$ which satisfies (\star_m) as follows. Let \mathcal{C}_{m-1} and \mathcal{C}_m be the cokernels of

$$(2.5) \quad \pi_*(\mathcal{E}(m-1)) \rightarrow (V_{m-1})_T, \quad \pi_*(\mathcal{E}(m)) \rightarrow (V_m)_T$$

induced by ι respectively. Since ι satisfies (\star_m) , the morphisms (2.5) are injective and \mathcal{C}_{m-1} and \mathcal{C}_m are locally free of rank $mr+d$ and $(m+1)r+d$ respectively. By the universal properties of Grassmannians, we obtain a morphism $\psi : T \rightarrow G_{m-1} \times G_m$, which satisfies $\psi^*(\text{pr}_1^* \mathcal{Q}_{m-1}) = \mathcal{C}_{m-1}$ and $\psi^*(\text{pr}_2^* \mathcal{Q}_m) = \mathcal{C}_m$.

Applying Proposition 2.6 to $\mathcal{E}(m) \rightarrow V_{\mathbb{P}^1 \times T}(m)$, we obtain

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi^*(\pi_* \mathcal{E}(m-1))(-1) & \longrightarrow & (V_{m-1})_{\mathbb{P}^1 \times T}(-1) & \longrightarrow & \pi^* \mathcal{C}_{m-1}(-1) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \pi^*(\pi_* \mathcal{E}(m)) & \longrightarrow & (V_m)_{\mathbb{P}^1 \times T} & \longrightarrow & \pi^* \mathcal{C}_m \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{E}(m) & \longrightarrow & V_{\mathbb{P}^1 \times T}(m) & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Tensoring $\mathcal{O}_{\mathbb{P}^1}(1)$ and pushing forward by π , we see that

$$\psi^*(p_m \circ j_m \circ i_{m-1}) : \psi^*(\text{pr}_1^* \mathcal{K}_{m-1}) = \pi_*(\mathcal{E}(m-1)) \rightarrow \mathcal{C}_m \otimes H = \psi^*(\text{pr}_2^* \mathcal{Q}_m) \otimes H$$

is zero. Hence ψ factors through R_m since R_m is the locus where $p_m \circ j_m \circ i_{m-1}$ vanishes. The cokernels \mathcal{C}_{m-1} and \mathcal{C}_m does not depend on the choice of the representative $\mathcal{E} \rightarrow V_{\mathbb{P}^1 \times T}$ of the equivalence class $[\mathcal{E} \rightarrow V_{\mathbb{P}^1 \times T}]$. Thus the morphism $\psi : T \rightarrow R_m$ is defined for the equivalence class $[\mathcal{E} \rightarrow V_{\mathbb{P}^1 \times T}]$. \square

3. SMOOTHNESS AND IRREDUCIBILITY OF R_m

In this section, we prove the rest part of Theorem 1.3,

First, we show that R_m is smooth and irreducible for any $m \geq \lceil d/s \rceil$. The proof is similar to that of Theorem 2.1 in [St], where Strømme proved the smoothness and irreducibility of R . However, there is a slight difference. In the proof, Strømme used the universal quotient sheaf \mathcal{B} on $\mathbb{P}^1 \times R$, but we use not the quotient \mathcal{Q} but \mathcal{K} in the diagram (2.4).

Proposition 3.1. *For $m \geq \lceil d/s \rceil$, R_m is smooth and irreducible.*

Proof. Let $M_0 := k^{\oplus s+d}$, $M_{-1} := k^{\oplus d}$ be vector spaces of dimensions $s+d, d$ respectively. Let W be the vector space

$$W = \text{Hom}_{\mathbb{P}^1}(M_0 \otimes \mathcal{O}_{\mathbb{P}^1}, M_{-1} \otimes \mathcal{O}_{\mathbb{P}^1}(1)) \times \text{Hom}(M_0, V)$$

and let $\overline{X} = \text{Spec}(\text{Sym } W^\vee)$ be the associated affine space. On $\mathbb{P}^1 \times \overline{X}$, we have a tautological diagram

$$\begin{array}{ccc} (M_0)_{\mathbb{P}^1 \times \overline{X}} & \xrightarrow{\nu} & (M_{-1})_{\mathbb{P}^1 \times \overline{X}}(1) \\ \downarrow \mu & & \\ V_{\mathbb{P}^1 \times \overline{X}} & & \end{array}$$

Let $X_m \subset \overline{X}$ be the open subset defined by the following three conditions; for each $x \in X_m$,

- 1) $\nu|_{\mathbb{P}^1 \times \{x\}}$ is surjective,
- 2) $H^1((\ker \nu)|_{\mathbb{P}^1 \times \{x\}} \otimes \mathcal{O}_{\mathbb{P}^1}(m-1)) = 0$,
- 3) the induced map $H^0((\ker \nu)|_{\mathbb{P}^1 \times \{x\}} \otimes \mathcal{O}_{\mathbb{P}^1}(m)) \rightarrow V_m$ is injective.

By 1), $(\ker \nu)|_{\mathbb{P}^1 \times \{x\}}$ is locally free of rank s and degree $-d$ for $x \in X_m$. Set $\tilde{\mathcal{K}} = (\ker \nu)|_{\mathbb{P}^1 \times X_m}$. By 1) - 3),

$$\tilde{\mathcal{K}} \hookrightarrow (M_0)_{\mathbb{P}^1 \times X_m} \xrightarrow{\mu|_{\mathbb{P}^1 \times X_m}} V_{\mathbb{P}^1 \times X_m}$$

satisfies condition (\star_m) . By Proposition 2.10, we have a morphism $g : X_m \rightarrow R_m$.

For $z \in R_m$, we can write $\mathcal{K}|_{\mathbb{P}^1 \times \{z\}} \cong \bigoplus_{i=1}^s \mathcal{O}(-a_i)$ for some $a_i \in \mathbb{Z}$ by Grothendieck. Since $\mathcal{K} \rightarrow V_{\mathbb{P}^1 \times R_m}$ satisfies condition (\star_m) by Lemma 2.8, $H^0(\mathcal{K}|_{\mathbb{P}^1 \times \{z\}}(m)) \rightarrow H^0(V_{\mathbb{P}^1} \otimes (m))$ is injective. Hence all a_i are nonnegative. Thus we can apply Proposition 2.6 to \mathcal{K}^\vee , and we have an exact sequence

$$(3.1) \quad 0 \rightarrow \pi^* \pi_* (\mathcal{K}^\vee(-1))(-1) \rightarrow \pi^* \pi_* (\mathcal{K}^\vee) \rightarrow \mathcal{K}^\vee \rightarrow 0$$

for $\pi = \pi_{R_m}$. Since $\mathcal{K}^\vee|_{\mathbb{P}^1 \times \{z\}}$ is locally free of rank s and degree d for $z \in R_m$, $\pi_*(\mathcal{K}^\vee)$ and $\pi_*(\mathcal{K}^\vee(-1))$ are locally free of ranks $s+d$ and d respectively. Let $Y_0 \rightarrow R_m$ (resp. $Y_{-1} \rightarrow R_m$) be the principal $\text{GL}(s+d)$ -bundle (resp. $\text{GL}(d)$ -bundle) associated to $\pi_*(\mathcal{K}^\vee)$ (resp. $\pi_*(\mathcal{K}^\vee(-1))$) (see [St, Section 2] for principal GL-bundles).

Pushing forward the dual sequence of

$$(3.2) \quad 0 \rightarrow \tilde{\mathcal{K}} \rightarrow (M_0)_{\mathbb{P}^1 \times X_m} \xrightarrow{\nu} (M_{-1})_{\mathbb{P}^1 \times X_m}(1) \rightarrow 0$$

by π_{X_m} , we have an isomorphism $(M_0^\vee)_{X_m} \rightarrow \pi_{X_m*}(\tilde{\mathcal{K}}^\vee)$. By this isomorphism, we can identify the dual of (3.2) with the exact sequence

$$0 \rightarrow \pi_{X_m}^* \pi_{X_m*}(\tilde{\mathcal{K}}^\vee(-1))(-1) \rightarrow \pi_{X_m}^* \pi_{X_m*}(\tilde{\mathcal{K}}^\vee) \rightarrow \tilde{\mathcal{K}}^\vee \rightarrow 0$$

obtained by applying Proposition 2.6 to $\tilde{\mathcal{K}}^\vee$. In particular, we have $(M_{-1}^\vee)_{X_m} \cong \pi_{X_m*}(\tilde{\mathcal{K}}^\vee(-1))$. By the construction of $g : X_m \rightarrow R_m$, it holds that $(\text{id}_{\mathbb{P}^1} \times g)^* \mathcal{K} = \tilde{\mathcal{K}}$. Hence we have

$$g^* \pi_*(\mathcal{K}^\vee) = \pi_{X_m*}(\tilde{\mathcal{K}}^\vee) \cong (M_0^\vee)_{X_m} = \mathcal{O}_{X_m}^{\oplus s+d}.$$

Similarly, we have $g^*\pi_*(\mathcal{K}^\vee(-1)) \cong (M_{-1}^\vee)_{X_m} = \mathcal{O}_{X_m}^{\oplus d}$. By the universal property of principal GL-bundles, g factors through $Y := Y_0 \times_{R_m} Y_{-1}$ as follows.

$$\begin{array}{ccc} X_m & \xrightarrow{\sigma} & Y = Y_0 \times_{R_m} Y_{-1} \\ & \searrow g & \downarrow \rho \\ & & R_m \end{array}$$

We show that σ is an isomorphism. Since Y_0 and Y_{-1} are principal GL-bundles, we have isomorphisms on Y

$$(3.3) \quad \rho^*\pi_*(\mathcal{K}^\vee) \xrightarrow{\sim} \mathcal{O}_Y^{\oplus s+d} = (M_0^\vee)_Y, \quad \rho^*\pi_*(\mathcal{K}^\vee(-1)) \xrightarrow{\sim} \mathcal{O}_Y^{\oplus d} = (M_{-1}^\vee)_Y.$$

Pulling back (3.1) by $\text{id}_{\mathbb{P}^1} \times \rho$, composing with the isomorphisms in (3.3), and taking the dual, we obtain a diagram on $\mathbb{P}^1 \times Y$

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\text{id}_{\mathbb{P}^1} \times \rho)^*\mathcal{K} & \longrightarrow & (M_0)_{\mathbb{P}^1 \times Y} & \longrightarrow & (M_{-1})_{\mathbb{P}^1 \times Y}(1) \longrightarrow 0 \\ & & \searrow & & & & \\ & & & & V_{\mathbb{P}^1 \times Y} & & \end{array}$$

where $(\text{id}_{\mathbb{P}^1} \times \rho)^*\mathcal{K} \rightarrow V_{\mathbb{P}^1 \times Y}$ is the pullback of the universal morphism $\mathcal{K} \rightarrow V_{\mathbb{P}^1 \times R_m}$ by $\text{id}_{\mathbb{P}^1} \times \rho$. Then there exists a unique lifting $(M_0)_{\mathbb{P}^1 \times Y} \rightarrow V_{\mathbb{P}^1 \times Y}$, which induces a morphism $u : Y \rightarrow X_m$. By construction, u is the inverse of σ .

Thus R_m is smooth since so are $Y \cong X_m$ and $\rho : Y \rightarrow R_m$. The irreducibility of R_m follows from that of X_m . \square

Proof of Theorem 1.3. By Propositions 2.10, 3.1, it remains to construct a birational map $\tilde{g}_m : R \dashrightarrow R_m$ for each $m \geq \lceil d/s \rceil$.

By construction, $\iota : \mathcal{E} \rightarrow V_{\mathbb{P}^1}$ in Remark 2.2 satisfies condition (\star_m) for any $m \geq \lceil d/s \rceil$. Hence R_m is not empty for any $m \geq \lceil d/s \rceil$.

Let $X_m \subset \overline{X}$ be the open subset in the proof of Proposition 3.1. Since $g : X_m \rightarrow R_m$ in the proof of Proposition 3.1 is surjective, X_m is nonempty for $m \geq \lceil d/s \rceil$. Hence $X_m \cap X_d$ is also a nonempty open subset of \overline{X} . This means that both R_m and R_d contain the set

$$\{[\mathcal{E} \rightarrow V_{\mathbb{P}^1}] \mid [\mathcal{E} \rightarrow V_{\mathbb{P}^1}] \text{ satisfies conditions } (\star_m) \text{ and } (\star_d)\}$$

as a nonempty open subset. Hence there exists a natural birational map $R_d \dashrightarrow R_m$. Then the composition of $(g_{d-1}, g_d) : R \rightarrow R_d$ and $R_d \dashrightarrow R_m$, which we denote by \tilde{g}_m , is birational by Theorem 2.9 and we finish the proof of Theorem 1.3. \square

Remark 3.2. By definition, \tilde{g}_m maps a general point $z \in R$ to $[\mathcal{A}|_{\mathbb{P}^1 \times \{z\}} \rightarrow V_{\mathbb{P}^1}] \in R_m$. Hence $\tilde{g}_m : R \dashrightarrow R_m \subset G_{m-1} \times G_m$ is nothing but (g_{m-1}, g_m) .

4. PROJECTIONS TO GRASSMANNIANS

In the previous sections, we consider Grassmannians of *subspace*. In this section, we consider Grassmannians of *quotient spaces*: For a vector space E , we denote by $\text{Gr}(E, r)$ the Grassmannian of r -dimensional quotient spaces of E . More generally, for a coherent sheaf \mathcal{E} on a noetherian scheme S , we set a scheme $\text{Gr}(\mathcal{E}, r)$ over S by

$$\text{Gr}(\mathcal{E}, r) := \text{Quot}_{\mathcal{E}/S/S}^{r, \mathcal{O}_S},$$

which parametrizes locally free quotient sheaves of $\varphi^*\mathcal{E}$ of rank r for each $\varphi : T \rightarrow S$ (see [Gr], [Ni, 5.1.5]). In particular, the fiber of $\mathrm{Gr}(\mathcal{E}, r) \rightarrow S$ over $s \in S$ is the Grassmannian $\mathrm{Gr}(\mathcal{E} \otimes k(s), r)$. If \mathcal{E} is locally free of rank n , we call $\mathrm{Gr}(\mathcal{E}, r) \rightarrow S$ a $\mathrm{Gr}(n, r)$ -*bundle* over S .

In this section, we study the projections $\mathrm{pr}_1 : R_m \rightarrow G_{m-1}$ and $\mathrm{pr}_2 : R_m \rightarrow G_m$. Throughout this section, we assume that $d \geq 1$.

For each $m \geq 0$, $j_m : V_{m-1} \rightarrow V_m \otimes H$ induces a linear map

$$k_m : V_{m-1} \otimes H^\vee \rightarrow V_m.$$

Then we have a morphism

$$k_m \circ i'_{m-1} : \mathcal{K}_{m-1} \otimes H^\vee \xrightarrow{i'_{m-1}} (V_{m-1})_{G_{m-1}} \otimes H^\vee \xrightarrow{k_m} (V_m)_{G_{m-1}}$$

of locally free sheaves on G_{m-1} for $m \geq \lceil d/s \rceil$, where i'_{m-1} is induced from the natural inclusion $\mathcal{K}_{m-1} \hookrightarrow (V_{m-1})_{G_{m-1}}$.

Lemma 4.1. *For $m \geq \lceil d/s \rceil$, the projection $\mathrm{pr}_1 : R_m \rightarrow G_{m-1}$ is isomorphic to $\mathrm{Gr}(\mathrm{coker}(k_m \circ i'_{m-1}), (m+1)r + d) \rightarrow G_{m-1}$ over G_{m-1} .*

Proof. Set $\mathbb{G}_m = \mathrm{Gr}(\mathrm{coker}(k_m \circ i'_{m-1}), (m+1)r + d)$. By the natural morphism $(V_m)_{G_{m-1}} \rightarrow \mathrm{coker}(k_m \circ i'_{m-1})$, we have a closed embedding

$$\mathbb{G}_m \hookrightarrow \mathrm{Gr}((V_m)_{G_{m-1}}, (m+1)r + d) = G_{m-1} \times G_m$$

over G_{m-1} since $\mathrm{Gr}(V_m, (m+1)r + d) = \mathrm{Gr}((m+1)s - d, V_m) = G_m$.

By Definition 2.7, $R_m \subset G_{m-1} \times G_m$ is also defined as the closed subscheme of the vanishing of $p'_m \circ k_m \circ i'_{m-1}$ for

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{pr}_1^* \mathcal{K}_{m-1} \otimes H^\vee & \xrightarrow{i'_{m-1}} & (V_{m-1})_{G_{m-1} \times G_m} \otimes H^\vee & \longrightarrow & \mathrm{pr}_1^* \mathcal{Q}_{m-1} \otimes H^\vee \longrightarrow 0 \\ & & & & \downarrow k_m & & \\ 0 & \longrightarrow & \mathrm{pr}_2^* \mathcal{K}_m & \longrightarrow & (V_m)_{G_{m-1} \times G_m} & \xrightarrow{p'_m} & \mathrm{pr}_2^* \mathcal{Q}_m \longrightarrow 0, \end{array}$$

where we use the same notation i'_{m-1} for the pullback of i'_{m-1} on G_{m-1} by pr_1 . By the construction of the embedding $\mathbb{G}_m \hookrightarrow G_{m-1} \times G_m$, the restriction of p'_m on \mathbb{G}_m factors through

$$(V_m)_{G_{m-1} \times G_m}|_{\mathbb{G}_m} = \mathrm{pr}_1^*(V_m)_{G_{m-1}}|_{\mathbb{G}_m} \rightarrow \mathrm{pr}_1^* \mathrm{coker}(k_m \circ i'_{m-1})|_{\mathbb{G}_m}.$$

Thus $p'_m \circ k_m \circ i'_{m-1}$ vanishes on \mathbb{G}_m , that is, \mathbb{G}_m is a closed subscheme of R_m .

On the other hand, the restriction of p'_m on R_m factors through

$$(V_m)_{G_{m-1} \times G_m}|_{R_m} \rightarrow \mathrm{pr}_1^* \mathrm{coker}(k_m \circ i'_{m-1})|_{R_m}$$

since $p'_m \circ k_m \circ i'_{m-1}$ vanishes on R_m . Since \mathcal{Q}_m is locally free of rank $(m+1)r + d$, the induced surjection $\mathrm{pr}_1^* \mathrm{coker}(k_m \circ i'_{m-1})|_{R_m} \twoheadrightarrow \mathrm{pr}_2^* \mathcal{Q}_m|_{R_m}$ gives a morphism $R_m \rightarrow \mathbb{G}_m$ by the universal property of \mathbb{G}_m .

By construction, $\mathbb{G}_m \hookrightarrow R_m$ and $R_m \rightarrow \mathbb{G}_m$ are inverses of each other. Hence R_m coincides with \mathbb{G}_m as a closed subscheme of $G_{m-1} \times G_m$, and this lemma follows. \square

On G_m , we have a morphism

$$p_m \circ j_m : (V_{m-1})_{G_m} \xrightarrow{j_m} (V_m)_{G_m} \otimes H \xrightarrow{p_m} \mathcal{Q}_m \otimes H.$$

Let $j_m^\vee \circ p_m^\vee : (\mathcal{Q}_m \otimes H)^\vee \rightarrow (V_{m-1})_{G_m}^\vee$ be the dual of $p_m \circ j_m$.

Lemma 4.2. *For $m \geq \lceil d/s \rceil$, the projection $\text{pr}_2 : R_m \rightarrow G_m$ is isomorphic to $\text{Gr}(\text{coker}(j_m^\vee \circ p_m^\vee), ms - d) \rightarrow G_m$ over G_m .*

Proof. By the natural morphism $(V_{m-1})_{G_m}^\vee \twoheadrightarrow \text{coker}(j_m^\vee \circ p_m^\vee)$, we have an embedding

$$\text{Gr}(\text{coker}(j_m^\vee \circ p_m^\vee), ms - d) \hookrightarrow \text{Gr}((V_{m-1})_{G_m}^\vee, ms - d) = G_{m-1} \times G_m$$

over G_m since $\text{Gr}((V_{m-1})^\vee, ms - d) = \text{Gr}(ms - d, V_{m-1}) = G_{m-1}$.

Since $R_m \subset G_{m-1} \times G_m$ is also defined as the closed subscheme of the vanishing of $i_{m-1}^\vee \circ j_m^\vee \circ p_m^\vee$ for

$$\begin{array}{ccccccc} 0 & \longleftarrow & \text{pr}_1^* \mathcal{K}_{m-1}^\vee & \xleftarrow{i_{m-1}^\vee} & (V_{m-1})_{G_{m-1} \times G_m}^\vee & \xleftarrow{\quad} & \text{pr}_1^* \mathcal{Q}_{m-1}^\vee \longleftarrow 0 \\ & & & & \uparrow j_m^\vee & & \\ 0 & \longleftarrow & (\text{pr}_2^* \mathcal{K}_m \otimes H)^\vee & \longleftarrow & (V_m \otimes H)_{G_{m-1} \times G_m}^\vee & \xleftarrow{p_m^\vee} & (\text{pr}_2^* \mathcal{Q}_m \otimes H)^\vee \longleftarrow 0, \end{array}$$

we can show that R_m coincides with $\text{Gr}(\text{coker}(j_m^\vee \circ p_m^\vee), ms - d)$ as a closed subscheme of $G_{m-1} \times G_m$ by an argument similar to that in the proof of Lemma 4.1. We leave the detail to the reader. \square

4.1. Stratification of pr_2 . In this subsection, we consider a stratification of $\text{pr}_2 : R_m \rightarrow G_m$.

Definition 4.3. For $m \geq \lceil d/s \rceil$ and $i \in \mathbb{Z}$, we define a closed subscheme X_m^i of G_m to be the $mr + d - i$ -th degeneracy locus of $j_m^\vee \circ p_m^\vee : (\mathcal{Q}_m \otimes H)^\vee \rightarrow (V_{m-1})_{G_m}^\vee$ (see [ACGH, Chapter 2] for degeneracy loci).

Lemma 4.4. *For $m \geq \lceil d/s \rceil$, X_m^0 is reduced, irreducible, Cohen-Macaulay, and $X_m^0 = \text{pr}_2(R_m)$. Moreover, $\text{pr}_2 : R_m \rightarrow X_m^0$ is an isomorphism over the non-empty open subset $X_m^0 \setminus X_m^1$. In particular, $X_m^0 \setminus X_m^1$ is smooth.*

Proof. By Theorem 1.3, Lemma 2.4, $\tilde{g}_m : R \dashrightarrow R_m$ and $g_m : R \dashrightarrow g_m(R) \subset G_m$ are birational. Since $g_m = \text{pr}_2 \circ \tilde{g}_m$ holds by Remark 3.2, $\text{pr}_2 : R_m \rightarrow \text{pr}_2(R_m) \subset G_m$ is birational as well.

We show that $\text{pr}_2(R_m) = X_m^0$ holds set-theoretically. By the definition of X_m^i , $\text{coker}(j_m^\vee \circ p_m^\vee)|_{X_m^i \setminus X_m^{i+1}}$ is local free of rank

$$\text{rank}(V_{m-1})_{G_m}^\vee - (mr + d - i) = ms - d + i.$$

Hence $\text{pr}_2(R_m) = X_m^0$ and $\text{pr}_2 : R_m \rightarrow G_m$ is a $\text{Gr}(ms - d + i, ms - d)$ -bundle over $X_m^i \setminus X_m^{i+1}$ for $i \geq 0$ by Lemma 4.2. In particular, X_m^0 is irreducible and $\text{pr}_2 : R_m \rightarrow X_m^0$ is an isomorphism over $X_m^0 \setminus X_m^1$. Since $\text{pr}_2 : R_m \rightarrow X_m^0$ is birational and $\text{pr}_2 : R_m \rightarrow G_m$ is not finite over X_m^1 , $X_m^0 \setminus X_m^1$ is non-empty.

Since $X_m^0 \subset G_m$ is the $mr + d$ -th degeneracy locus of $(\mathcal{Q}_m \otimes H)^\vee \rightarrow (V_{m-1})_{G_m}^\vee$, the expected codimension of X_m^0 in G_m is

$$(\text{rank}(V_{m-1})_{G_m}^\vee - (mr + d))(\text{rank}(\mathcal{Q}_m \otimes H)^\vee - (mr + d)).$$

Since $\dim X_m^0 = \dim R_m = nd + rs$, we can check that the expected codimension coincides with $\dim G_m - \dim X_m^0$. Hence X_m^0 is Cohen Macaulay by [ACGH, Chapter II, (4.1) Proposition].

Since $\text{pr}_2 : R_m \rightarrow X_m^0$ is an isomorphism over $X_m^0 \setminus X_m^1$, $X_m^0 \setminus X_m^1$ is smooth. Since X_m^0 is Cohen-Macaulay and reduced on the non-empty open subset $X_m^0 \setminus X_m^1$, X_m^0 is reduced. \square

Proposition 4.5. *For $m \geq \lceil d/s \rceil$, the following hold.*

- (1) *For $0 \leq i \leq \lfloor d/(m+1) \rfloor$, X_m^i is irreducible, Cohen-Macaulay, $X_m^i \setminus X_m^{i+1}$ is smooth, and $\dim X_m^i = n(d - (m+1)i) + (r+i)(s-i)$.*
- (2) *X_m^i is normal for $0 \leq i \leq \lfloor d/(m+1) \rfloor$.*
- (3) *$X_m^i = \emptyset$ for $i > \lfloor d/(m+1) \rfloor$.*
- (4) *$\text{pr}_2 : R_m \rightarrow G_m$ is a $\text{Gr}(ms - d + i, ms - d)$ -bundle over $X_m^i \setminus X_m^{i+1}$.*

In particular, $\text{pr}_2 : R_m \rightarrow X_m^0 \subset G_m$ is an isomorphism in codimension one.

Proof. We note that (4) is already shown in the second paragraph of the proof of Lemma 4.4,

First, we show (1). In this proof, we denote R, R_m, X_m^i by $R(s, d), R_m(s, d), X_m^i(s, d)$ respectively when we need to clarify s, d . Since

$$(m+1)s - d = (m+1)(s-i) - (d - (m+1)i),$$

we have a morphism

$$\text{pr}_2^i : R_m(s-i, d - (m+1)i) \rightarrow G_m = \text{Gr}((m+1)s - d, V_m)$$

as $\text{pr}_2 : R_m \rightarrow G_m$ for each $0 \leq i \leq \lfloor d/(m+1) \rfloor$.

Applying Lemma 4.4 to $R_m(s-i, d - (m+1)i)$, the image of pr_2^i is $X_m^0(s-i, d - (m+1)i)$, which is the $m(r+i) + (d - (m+1)i)$ -th degeneracy locus of $j_m^\vee \circ p_m^\vee : (\mathcal{Q}_m \otimes H)^\vee \rightarrow (V_{m-1})_{G_m}^\vee$. Since $m(r+i) + (d - (m+1)i) = mr + d - i$, $X_m^0(s-i, d - (m+1)i)$ coincides with $X_m^i(s, d)$. Similarly, $X_m^1(s-i, d - (m+1)i) = X_m^{i+1}(s, d)$ holds for each i . Applying Lemma 4.4 to $X_m^0(s-i, d - (m+1)i)$ and $X_m^1(s-i, d - (m+1)i)$, we obtain (1).

We show (3). It suffices to see that

$$X_m^{\lfloor d/(m+1) \rfloor + 1}(s, d) = X_m^1(s - \lfloor d/(m+1) \rfloor, d - (m+1)\lfloor d/(m+1) \rfloor)$$

is empty. By (4), $X_m^1(s - \lfloor d/(m+1) \rfloor, d - (m+1)\lfloor d/(m+1) \rfloor)$ is the image of the exceptional locus of $\text{pr}_2^{\lfloor d/(m+1) \rfloor}$. Hence it suffices to show that $\text{pr}_2^{\lfloor d/(m+1) \rfloor}$ is an embedding. Since $d - (m+1)\lfloor d/(m+1) \rfloor \leq m$,

$$R(s - \lfloor d/(m+1) \rfloor, d - (m+1)\lfloor d/(m+1) \rfloor) = R_m(s - \lfloor d/(m+1) \rfloor, d - (m+1)\lfloor d/(m+1) \rfloor)$$

holds by Theorem 2.9. Thus $\text{pr}_2^{\lfloor d/(m+1) \rfloor}$ is an embedding by Theorem 2.3 and (3) is shown.

To show (2), it suffices to see that X_m^0 is normal since $X_m^i(s, d) = X_m^0(s-i, d - (m+1)i)$.

By (1), (3), $X_m^0 \setminus X_m^1$ is smooth and $\dim X_m^1 \leq \dim X_m^0 - 2$. Hence X_m^0 is smooth in codimension one. Since X_m^0 is Cohen-Macaulay, X_m^0 is normal by Serre's criterion.

We show the last statement. If $m \geq d$, $X_m^1 = \emptyset$ by (3) since $1 > \lfloor d/(m+1) \rfloor$. Thus $\text{pr}_2 : R_m \rightarrow X_m^0$ is an isomorphism by (4).

If $\lceil d/s \rceil \leq m \leq d-1$, it holds that $\lfloor d/(m+1) \rfloor \geq 1$, hence we can compute the dimension of the exceptional locus $\text{pr}_2^{-1}(X_m^1) \subset R_m$ as

$$\begin{aligned} \dim \text{pr}_2^{-1}(X_m^1) &= \max_{1 \leq i \leq \lfloor d/(m+1) \rfloor} n(d - (m+1)i) + (r+i)(s-i) + i(ms-d) \\ &= nd + rs - (m+2)r - d - 1 \end{aligned}$$

by (1), (3), and (4). Since $d \geq 1$, $\dim \text{pr}_2^{-1}(X_m^1) \leq \dim R_m - 2$ holds. Hence $\text{pr}_2 : R_m \rightarrow Y_m^0 \subset G_m$ is an isomorphism in codimension one. \square

4.2. Stratification of pr_1 . In this subsection, we consider pr_1 . As we will see, $\text{pr}_1 : R_m \rightarrow G_{m-1}$ is birational for $m \geq \lceil d/s \rceil + 1$, and is not for $m = \lceil d/s \rceil$. Hence we study $\text{pr}_1 : R_m \rightarrow G_{m-1}$ for $m \geq \lceil d/s \rceil + 1$, namely, $\text{pr}_1 : R_{m+1} \rightarrow G_m$ for $m \geq \lceil d/s \rceil$ first. We study $\text{pr}_1 : R_{\lceil d/s \rceil} \rightarrow G_{\lceil d/s \rceil - 1}$ next.

Proposition 4.6. *For $m \geq \lceil d/s \rceil$, it holds that*

- (a) $\text{pr}_1(R_{m+1}) = X_m^0 \subset G_m$,
- (b) $\text{pr}_1 : R_{m+1} \rightarrow X_m^0$ is a $\text{Gr}((m+2)r + d + i, (m+2)r + d)$ -bundle over $X_m^i \setminus X_m^{i+1}$ for each $0 \leq i \leq \lfloor d/(m+1) \rfloor$.

In particular, $\text{pr}_1 : R_{m+1} \rightarrow X_m^0$ is an isomorphism in codimension one for $m \geq \lfloor d/s \rfloor + 1$. On the other hand, $\text{pr}_1 : R_{(d/s)+1} \rightarrow X_{d/s}^0 = G_{d/s}$ contracts a divisor if $d/s \in \mathbb{N}$.

Proof. Since g_m coincides with $\text{pr}_1 \circ \tilde{g}_{m+1} : R \dashrightarrow R_{m+1} \rightarrow \text{pr}_1(R_{m+1}) \subset G_m$ and g_m, \tilde{g}_{m+1} are birational, $\text{pr}_1 : R_{m+1} \rightarrow \text{pr}_1(R_{m+1}) \subset G_m$ is birational for $m \geq \lceil d/s \rceil$.

For $i \geq 0$, let $Y_m^i \subset G_m$ be the $(m+2)s - d - i$ -th degeneracy locus of $k_{m+1} \circ i'_m : \mathcal{K}_m \otimes H^\vee \rightarrow (V_{m+1})_{G_m}$. By the same argument as Lemma 4.4, Proposition 4.5, we can show that $\text{pr}_1(R_{m+1}) = Y_m^0$ and

- (1)' For $0 \leq i \leq \lfloor d/(m+1) \rfloor$, Y_m^i is irreducible, Cohen-Macaulay, $Y_m^i \setminus Y_m^{i+1}$ is smooth, and $\dim Y_m^i = n(d - (m+1)i) + (r+i)(s-i)$.
- (2)' Y_m^i is normal for $0 \leq i \leq \lfloor d/(m+1) \rfloor$.
- (3)' $Y_m^i = \emptyset$ for $i > \lfloor d/(m+1) \rfloor$.
- (4)' $\text{pr}_1 : R_{m+1} \rightarrow G_m$ is a $\text{Gr}((m+2)r + d + i, (m+2)r + d)$ -bundle over $Y_m^i \setminus Y_m^{i+1}$.

Hence it suffices to show $X_m^i = Y_m^i$ for (a), (b).

Since $g_m = \text{pr}_1 \circ \tilde{g}_{m+1} = \text{pr}_2 \circ \tilde{g}_m : R \dashrightarrow G_m$, we have $\overline{g_m(R)} = \text{pr}_1(R_{m+1}) = \text{pr}_2(R_m)$, where $\overline{g_m(R)}$ is the closure of $g_m(R)$. Since $\text{pr}_1(R_{m+1}) = Y_m^0$ and $\text{pr}_2(R_m) = X_m^0$, it holds that $X_m^0 = Y_m^0$.

For $0 \leq i \leq \lfloor d/(m+1) \rfloor$, we denote X_m^i, Y_m^i by $X_m^i(s, d), Y_m^i(s, d)$ respectively to clarify s, d . As in the proof of Proposition 4.6, it holds that $X_m^i(s, d) = X_m^0(s - i, d - (m+1)i)$. By a similar argument, $Y_m^i(s, d) = Y_m^0(s - i, d - (m+1)i)$ holds. Hence we have $X_m^i(s, d) = X_m^0(s - i, d - (m+1)i) = Y_m^0(s - i, d - (m+1)i) = Y_m^i(s, d)$. Thus (a), (b) are proved.

If $m \geq d$, $\text{pr}_1 : R_{m+1} \rightarrow Y_m^0$ is an isomorphism by (3)' and (4)'. For $\lfloor d/s \rfloor + 1 \leq m \leq d - 1$, the dimension of the exceptional locus $\text{pr}_1^{-1}(Y_m^1) \subset R_{m+1}$ of pr_1 is

$$nd + rs - (sm - d + 1) = \dim R_{m+1} - (sm - d + 1).$$

by (1)', (3)', and (4)'. Hence $\text{pr}_1 : R_{m+1} \rightarrow Y_m^0$ is an isomorphism in codimension one for $\lfloor d/s \rfloor + 1 \leq m \leq d - 1$. On the other hand, $\text{pr}_1 : R_{(d/s)+1} \rightarrow Y_{d/s}^0$ contracts a divisor if $d/s \in \mathbb{N}$. In this case, $Y_{d/s}^0 = G_{d/s}$ holds since $\dim Y_{d/s}^0 = nd + rs = \dim G_{d/s}$. \square

In Definition 4.3 and the proof of Proposition 4.6, we defined $X_m^i = Y_m^i \subset G_m$ for $m \geq \lceil d/s \rceil$. For $m = \lceil d/s \rceil - 1$, we define $X_{\lceil d/s \rceil - 1}^i \subset G_{\lceil d/s \rceil - 1}$ in a slightly different manner.

Definition 4.7. Set $l = \lceil d/s \rceil s - d$. For $i \geq 0$, we define a closed subscheme $X_{\lceil d/s \rceil - 1}^i$ of $G_{\lceil d/s \rceil - 1} = \text{Gr}(l, V_{\lceil d/s \rceil - 1})$ to be the $2l - i$ -th degeneracy locus of

$$k_{\lceil d/s \rceil} \circ i'_{\lceil d/s \rceil - 1} : \mathcal{K}_{\lceil d/s \rceil - 1} \otimes H^\vee \rightarrow (V_{\lceil d/s \rceil})_{G_{\lceil d/s \rceil - 1}}.$$

Since $\text{rank } \mathcal{K}_{[d/s]-1} \otimes H^\vee = 2l$, it holds that $X_{[d/s]-1}^0 = G_{[d/s]-1}$.

Proposition 4.8. *In the case $m = [d/s] - 1$, the following hold.*

- (1) *For $0 \leq i \leq l - [l/[d/s]]$, $X_{[d/s]-1}^i$ is irreducible, normal, Cohen-Macaulay, $X_{[d/s]-1}^i \setminus X_{[d/s]-1}^{i+1}$ is smooth, and $\dim X_{[d/s]-1}^i = n(([d/s] - 1)l - [d/s]i) + (n - l + i)(l - i)$.*
- (2) *$X_{[d/s]-1}^i = \emptyset$ for $i > l - [l/[d/s]]$.*
- (3) *$\text{pr}_1 : R_{[d/s]} \rightarrow G_{[d/s]-1}$ is a $\text{Gr}(([d/s] + 1)n - 2l + i, ([d/s] + 1)r + d)$ -bundle over $X_{[d/s]-1}^i \setminus X_{[d/s]-1}^{i+1}$ for $0 \leq i \leq l - [l/[d/s]]$.*

In particular, $\text{pr}_1(R_{[d/s]}) = X_{[d/s]-1}^0 = G_{[d/s]-1}$ holds and any fiber of $\text{pr}_1 : R_{[d/s]} \rightarrow G_{[d/s]-1}$ is a Grassmannian, which is not a point.

Proof. Since

$$[d/s]s - d = l = [d/s]l - ([d/s] - 1)l,$$

$G_{[d/s]-1} = \text{Gr}(l, V_{[d/s]-1})$ does not change by replacing s, d with $\bar{s} := l, \bar{d} := ([d/s] - 1)l$. Since $m = [d/s] - 1 \geq [\bar{d}/\bar{s}]$, we can define $X_{[d/s]-1}^i(\bar{s}, \bar{d}) \subset G_{[d/s]-1}$ by Definition 4.3. Since $X_{[d/s]-1}^i(\bar{s}, \bar{d}) = Y_{[d/s]-1}^i(\bar{s}, \bar{d})$ is the $([d/s] + 1)\bar{s} - \bar{d} - i$ -th degeneracy locus of

$$k_{[d/s]} \circ i'_{[d/s]-1} : \mathcal{K}_{[d/s]-1} \otimes H^\vee \rightarrow (V_{[d/s]})_{G_{[d/s]-1}}$$

and $([d/s] + 1)\bar{s} - \bar{d} - i = 2l - i$, $X_{[d/s]-1}^i$ in Definition 4.7 coincides with $X_{[d/s]-1}^i(\bar{s}, \bar{d})$.

Applying (1), (2), (3) in Proposition 4.5 to $X_{[d/s]-1}^i(\bar{s}, \bar{d})$, we have (1), (2) in this proposition.

By definition, the restriction of $\text{coker}(k_{[d/s]} \circ i'_{[d/s]-1})$ on $X_{[d/s]-1}^i \setminus X_{[d/s]-1}^{i+1}$ is locally free of rank $([d/s] + 1)n - 2l + i$. Hence (3) follows from Lemma 4.1.

For the last statement, it suffices to show that a general fiber of pr_1 is not a point. By (3), the general fiber of $\text{pr}_1 : R_{[d/s]} \rightarrow G_{[d/s]-1} = X_{[d/s]-1}^0$ is the Grassmannian $\text{Gr}(([d/s] + 1)n - 2l, ([d/s] + 1)r + d)$, which is not a point since

$$([d/s] + 1)n - 2l - ([d/s] + 1)r - d = s - l > 0$$

and $([d/s] + 1)r + d > 0$. \square

5. MOVABLE AND EFFECTIVE CONES OF R

In this section, we prove Theorems 1.5, 1.6. Throughout this section, we assume $d \geq 1$ and $0 \leq r \leq n - 2$.

Strømme defined line bundles α, β on R by $\alpha = c_1(\mathcal{B}_d) - c_1(\mathcal{B}_{d-1})$ and $\beta = c_1(\mathcal{B}_{d-1})$ for $\mathcal{B}_m = \pi_*(\mathcal{B}(m))$. For $m \geq 0$, there exists an exact sequence $0 \rightarrow \mathcal{B}_{m-1} \rightarrow \mathcal{B}_m^{\oplus 2} \rightarrow \mathcal{B}_{m+1} \rightarrow 0$ induced by $0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow 0$ (see [St, Lemma 5.1]). Hence it holds that $c_1(\mathcal{B}_m) = -(d - 1 - m)\alpha + \beta$ for $m \geq -1$.

Lemma 5.1. *For $[d/s] + 1 \leq m \leq d - 1$, the birational map $\tilde{g}_m : R \dashrightarrow R_m$ is an SQM of R . Under the identification of $N^1(R_m)_{\mathbb{R}}$ with $N^1(R)_{\mathbb{R}}$, it holds that*

$$\begin{aligned} \text{Nef}(R_m) &= \mathbb{R}_{\geq 0}c_1(\mathcal{B}_{m-1}) + \mathbb{R}_{\geq 0}c_1(\mathcal{B}_m) \\ &= \mathbb{R}_{\geq 0}(-(d - m)\alpha + \beta) + \mathbb{R}_{\geq 0}(-(d - 1 - m)\alpha + \beta), \end{aligned}$$

and $c_1(\mathcal{B}_{m-1})$ and $c_1(\mathcal{B}_m)$ are base point free on R_m .

Proof. Consider the following diagram

$$(5.1) \quad \begin{array}{ccc} R_{m+1} & \dashrightarrow & R_m \\ & \searrow \text{pr}_1 \quad \swarrow \text{pr}_2 & \\ & X_m^0 \subset G_m & \end{array}$$

for $\lfloor d/s \rfloor + 1 \leq m \leq d-1$. By Propositions 4.5, 4.6, the birational morphism pr_1, pr_2 in the diagram (5.1) are isomorphisms in codimension one. Hence $R_{m+1} \dashrightarrow R_m$ is also an isomorphism in codimension one. Since $\tilde{g}_m : R \dashrightarrow R_m$ is decomposed as

$$R \xrightarrow[\tilde{g}_d]{\sim} R_d \dashrightarrow R_{d-1} \dashrightarrow \cdots \dashrightarrow R_{m+1} \dashrightarrow R_m,$$

$\tilde{g}_m : R \dashrightarrow R_m$ is an isomorphism in codimension one as well. Since R_m is smooth by Proposition 3.1, $\tilde{g}_m : R \dashrightarrow R_m$ is an SQM of R .

By Propositions 4.5, 4.6, and 4.8, $\text{pr}_1 : R_m \rightarrow G_{m-1}$ and $\text{pr}_2 : R_m \rightarrow G_m$ are not finite morphisms for $\lfloor d/s \rfloor + 1 \leq m \leq d-1$. Hence $\text{Nef}(R_m)$ is spanned by $\text{pr}_1^* c_1(\mathcal{Q}_{m-1})$ and $\text{pr}_2^* c_1(\mathcal{Q}_m)$ since $c_1(\mathcal{Q}_{m-1})$ and $c_1(\mathcal{Q}_m)$ are ample line bundles on the Grassmannians G_{m-1} and G_m respectively. By the definition of $g_m : R \dashrightarrow G_m$, $c_1(\mathcal{B}_m)$ is the pullback of $c_1(\mathcal{Q}_m)$ by g_m (note that g_m is defined outside codimension two locus since so is $\tilde{g}_m : R \dashrightarrow R_m$). Hence, $\text{pr}_2^* c_1(\mathcal{Q}_m) = c_1(\mathcal{B}_m)$ holds under the identification of $N^1(R_m)_{\mathbb{R}}$ with $N^1(R)_{\mathbb{R}}$. Similarly, $\text{pr}_1^* c_1(\mathcal{Q}_{m-1}) = c_1(\mathcal{B}_{m-1})$ holds. Hence $\text{Nef}(R_m)$ is spanned by the two base point free line bundles $c_1(\mathcal{B}_{m-1})$ and $c_1(\mathcal{B}_m)$. \square

Remark 5.2. By Lemma 5.1, the Picard number of R_m is two for $\lfloor d/s \rfloor + 1 \leq m \leq d-1$. Hence pr_1 and pr_2 in the diagram (5.1) are small contractions for $\lfloor d/s \rfloor + 1 \leq m \leq d-1$ since X_m^0 is normal by Proposition 4.5.

Lemma 5.3. *The morphism $\text{pr}_1 : R_{\lfloor d/s \rfloor + 1} \rightarrow G_{\lfloor d/s \rfloor}$ is a fiber type contraction (resp. a divisorial contraction) if $d/s \notin \mathbb{N}$ (resp. $d/s \in \mathbb{N}$). Furthermore, $\mathbb{R}_{\geq 0} c_1(\mathcal{B}_{\lfloor d/s \rfloor - 1})$ is an edge of $\text{Mov}(R)$, and $\mathbb{R}_{\geq 0} c_1(\mathcal{B}_{\lfloor d/s \rfloor - 1})$ is an edge of $\text{Eff}(R)$.*

Proof. First, assume $d/s \notin \mathbb{N}$. Since the Picard number of $R_{\lfloor d/s \rfloor + 1}$ is two, $\text{pr}_1 : R_{\lfloor d/s \rfloor + 1} \rightarrow G_{\lfloor d/s \rfloor}$ is a fiber type contraction by Propositions 4.8. Thus $\text{pr}_1^* c_1(\mathcal{Q}_{\lfloor d/s \rfloor - 1}) = c_1(\mathcal{B}_{\lfloor d/s \rfloor - 1})$ spans an edge of both $\text{Mov}(R_{\lfloor d/s \rfloor})$ and $\text{Eff}(R_{\lfloor d/s \rfloor})$. Since $\lfloor d/s \rfloor - 1 = \lfloor d/s \rfloor$ and $R_{\lfloor d/s \rfloor} = R_{\lfloor d/s \rfloor + 1}$ is an SQM of R , this lemma holds if $d/s \notin \mathbb{N}$.

Next, assume $d/s \in \mathbb{N}$. Since the Picard number of $R_{\lfloor d/s \rfloor + 1}$ is two, $\text{pr}_1 : R_{(d/s)+1} \rightarrow G_{d/s}$ is a divisorial contraction by Propositions 4.6. Hence $\text{pr}_1^* c_1(\mathcal{Q}_{d/s}) = c_1(\mathcal{B}_{d/s})$ spans an edge of $\text{Mov}(R_{(d/s)+1})$ and E spans an edge of $\text{Eff}(R_{(d/s)+1})$, where E is the contracted divisor of $\text{pr}_1 : R_{(d/s)+1} \rightarrow G_{d/s}$. To compute the class of E , we compare the canonical divisors on $R_{(d/s)+1}$ and $G_{d/s}$. By [St, Theorem 7.1 (ii)], it holds that $K_R = -(n + (2r + 2 - n)d)\alpha - (n - 2r)\beta$. On the other hand, $K_{G_{d/s}} = -n((d/s) + 1)c_1(\mathcal{Q}_{d/s})$ holds since $G_{d/s}$ is a Grassmannian. Hence we have

$$\begin{aligned} K_{R_{(d/s)+1}} - \text{pr}_1^* K_{G_{d/s}} &= -(n + (2r + 2 - n)d)\alpha - (n - 2r)\beta + n((d/s) + 1)c_1(\mathcal{B}_{d/s}) \\ &= ((nd/s) + 2r)c_1(\mathcal{B}_{(d/s)-1}). \end{aligned}$$

Thus the class of E is a positive multiple of $c_1(\mathcal{B}_{(d/s)-1}) = c_1(\mathcal{B}_{\lfloor d/s \rfloor - 1})$ and this class spans an edge of $\text{Eff}(R_{(d/s)+1}) = \text{Eff}(R)$. \square

To find the other edges of $\text{Mov}(R)$ and $\text{Eff}(R)$, we recall the morphism defined by $\alpha = c_1(\mathcal{B}_d) - c_1(\mathcal{B}_{d-1})$ (see [St, Lemma 6.4]). Taking the s -th exterior power of the universal inclusion $\mathcal{A} \hookrightarrow V_{\mathbb{P}^1 \times R}$, we have $\det \mathcal{A} \rightarrow \bigwedge^s V_{\mathbb{P}^1 \times R}$. By tensoring $\mathcal{O}_{\mathbb{P}^1}(d)$ and taking π_* , we obtain a nowhere vanishing monomorphism

$$\mathcal{O}_R(-\alpha) \rightarrow \bigwedge^s V_R \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d)).$$

This induces a morphism $f : R \rightarrow \mathbb{P}(\bigwedge^r V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)$.

Definition 5.4. We define $K_{s,r}^d \subset \mathbb{P}(\bigwedge^r V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)$ to be the image $f(R)$ with the reduced structure. The subvariety $K_{s,r}^d$ is called a quantum Grassmannian, and studied in [Ro], [SS], etc.

Lemma 5.5. *The quantum Grassmannian $K_{s,r}^d$ is normal and Cohen-Macaulay.*

Proof. By [SS, Corollary 17], the coordinate ring of $K_{s,r}^d \subset \mathbb{P}(\bigwedge^r V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)$ is normal and Cohen-Macaulay. Hence this lemma follows. \square

Lemma 5.6. *For $r = 0$, f is surjective, i.e., $K_{n,0}^d = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)$ holds, and $f : R \rightarrow \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)$ is a fiber type contraction. In particular, $\mathbb{R}_{\geq 0}\alpha$ is an edge of both $\text{Mov}(R)$ and $\text{Eff}(R)$.*

Proof. By the assumption $0 = r \leq n - 2$, we have $n \geq 2$. Hence $\dim R = nd > \dim \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee) = d$ holds. Thus to prove this lemma, it is enough to show that f is a contraction, that is, $f_*\mathcal{O}_R = \mathcal{O}_{\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)}$ holds. To show $f_*\mathcal{O}_R = \mathcal{O}_{\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)}$, it suffices to see that there exists an open subset $U \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)$ such that f is smooth with irreducible fibers over U since $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)$ is normal.

Set $U = \{[D] \in |\mathcal{O}_{\mathbb{P}^1}(d)| \mid D \text{ is reduced}\} \subset |\mathcal{O}_{\mathbb{P}^1}(d)| = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)$. Let $\mathcal{D} \subset \mathbb{P}^1 \times U$ be the universal divisor, that is, $\mathcal{D}|_{\mathbb{P}^1 \times \{[D]\}} = D$ holds for each $[D] \in U$. Then $(\pi_U)_*\mathcal{H}om(V_{\mathbb{P}^1 \times U}, \mathcal{O}_{\mathcal{D}})$ is a locally free sheaf on $|\mathcal{O}_{\mathbb{P}^1}(d)|$ of rank nd . Hence we obtain the vector bundle

$$\mathbb{V}_U((\pi_U)_*\mathcal{H}om(V_{\mathbb{P}^1 \times U}, \mathcal{O}_{\mathcal{D}})^\vee) \rightarrow U,$$

whose fiber over $[D] \in U$ is the affine space $\text{Hom}(V_{\mathbb{P}^1}, \mathcal{O}_D) \cong \mathbb{A}^{nd}$. We write a point in $\mathbb{V}_U((\pi_U)_*\mathcal{H}om(V_{\mathbb{P}^1 \times U}, \mathcal{O}_{\mathcal{D}})^\vee)$ as $([D], [V_{\mathbb{P}^1} \rightarrow \mathcal{O}_D])$. Set an open subset \tilde{U} by

$$\tilde{U} := \{([D], [V_{\mathbb{P}^1} \rightarrow \mathcal{O}_D]) \mid V_{\mathbb{P}^1} \rightarrow \mathcal{O}_D \text{ is surjective}\} \subset \mathbb{V}_U((\pi_U)_*\mathcal{H}om(V_{\mathbb{P}^1 \times U}, \mathcal{O}_{\mathcal{D}})^\vee).$$

By definition, f maps a point $[V_{\mathbb{P}^1} \rightarrow B] \in R$ to

$$\sum_{P \in \mathbb{P}^1} (\text{length}_{k(P)} B \otimes k(P)) P \in |\mathcal{O}_{\mathbb{P}^1}(d)| = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee).$$

We note that B is a torsion sheaf since $\text{rank } B = r = 0$. Hence for $[D] \in U$ and $[V_{\mathbb{P}^1} \rightarrow B] \in f^{-1}([D])$, it holds that $B \cong \mathcal{O}_D$ since D is reduced. Thus $(\pi_{f^{-1}(U)})_*\mathcal{H}om(\mathcal{B}, (\text{id}_{\mathbb{P}^1} \times f)^*\mathcal{O}_{\mathcal{D}})$ is a locally free sheaf on $f^{-1}(U)$ of rank d . Hence we obtain

$$\mathbb{V}_{f^{-1}(U)}((\pi_{f^{-1}(U)})_*\mathcal{H}om(\mathcal{B}, (\text{id}_{\mathbb{P}^1} \times f)^*\mathcal{O}_{\mathcal{D}})^\vee) \rightarrow f^{-1}(U),$$

whose fiber over $[V_{\mathbb{P}^1} \rightarrow B] \in f^{-1}(U)$ is the affine space $\text{Hom}(B, \mathcal{O}_D) \cong \mathbb{A}^d$, where $[D] = f([V_{\mathbb{P}^1} \rightarrow B])$. Set an open subset $\widetilde{f^{-1}(U)}$ by

$$\begin{aligned} \widetilde{f^{-1}(U)} &= \{([V_{\mathbb{P}^1} \rightarrow B], [B \rightarrow \mathcal{O}_D]) \mid B \rightarrow \mathcal{O}_D \text{ is an isomorphism}\} \\ &\subset \mathbb{V}_{f^{-1}(U)}((\pi_{f^{-1}(U)})_* \mathcal{H}om(\mathcal{B}, (\text{id}_{\mathbb{P}^1} \times f)^* \mathcal{O}_D)^\vee). \end{aligned}$$

Claim 5.7. *There exists an isomorphism $\tilde{f} : \widetilde{f^{-1}(U)} \rightarrow \tilde{U}$ such that*

$$\begin{array}{ccc} \widetilde{f^{-1}(U)} & \xrightarrow{\tilde{f}} & \tilde{U} \\ \downarrow & & \downarrow \\ f^{-1}(U) & \xrightarrow{f} & U \end{array}$$

is commutative.

Proof of Claim 5.7. We define \tilde{f} by mapping $([V_{\mathbb{P}^1} \xrightarrow{q} B], [B \xrightarrow{i} \mathcal{O}_D]) \in \widetilde{f^{-1}(U)}$ to $([D], [V_{\mathbb{P}^1} \xrightarrow{q} B \xrightarrow{i} \mathcal{O}_D]) \in \tilde{U}$. The converse $\tilde{U} \rightarrow \widetilde{f^{-1}(U)}$ is defined by mapping $([D], [V_{\mathbb{P}^1} \xrightarrow{\bar{q}} \mathcal{O}_D]) \in \tilde{U}$ to

$$([V_{\mathbb{P}^1} \xrightarrow{\bar{q}} \mathcal{O}_D], [\mathcal{O}_D \xrightarrow{\text{id}} \mathcal{O}_D]) \in \widetilde{f^{-1}(U)}.$$

□

Since $\widetilde{f^{-1}(U)} \rightarrow f^{-1}(U)$ and $\tilde{U} \rightarrow U$ are smooth morphisms with irreducible fibers, so is f by Claim 5.7 and this lemma is proved. □

Assume $r \geq 1$. Let R' be the Quot scheme parametrizing rank s and degree d quotient sheaves of the dual bundle $V_{\mathbb{P}^1}^\vee$ and let

$$(5.2) \quad 0 \rightarrow \mathcal{A}' \rightarrow V_{\mathbb{P}^1 \times R'}^\vee \rightarrow \mathcal{B}' \rightarrow 0$$

be the universal exact sequence on $\mathbb{P}^1 \times R'$. Let $R'^\circ \subset R'$ be the open subset corresponding to locally free quotient sheaves of $V_{\mathbb{P}^1}^\vee$. Taking the dual of the universal sequence (2.1) on $\mathbb{P}^1 \times R$, we have a sequence

$$(5.3) \quad 0 \rightarrow \mathcal{B}^\vee \rightarrow V_{\mathbb{P}^1 \times R}^\vee \rightarrow \mathcal{A}^\vee \rightarrow 0,$$

which is exact on $\mathbb{P}^1 \times R^\circ$. By the universal property of R' , we have a morphism $R^\circ \rightarrow R'$. Similarly, we have a morphism $R'^\circ \rightarrow R$. By these morphisms, we have an isomorphism $R^\circ \cong R'^\circ$. Under this isomorphism, the restriction of (5.3) on $\mathbb{P}^1 \times R^\circ$ coincides with that of (5.2) on $\mathbb{P}^1 \times R'^\circ$.

Lemma 5.8. *If $r = 1$, f is surjective, i.e., $K_{n-1,1}^d = \mathbb{P}(V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d)))^\vee$ holds, and $f : R \rightarrow \mathbb{P}(V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d)))^\vee$ is a divisorial contraction. In particular, $\mathbb{R}_{\geq 0}\alpha$ and $\mathbb{R}_{\geq 0}(2d\alpha - \beta)$ are edges of $\text{Mov}(R)$ and $\text{Eff}(R)$ respectively.*

Proof. When $r = 1$, $R' = \mathbb{P}(V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d)))^\vee$ by [St, Proposition 6.1 (i)]. By construction, the isomorphism $R^\circ \cong R'^\circ$ is nothing but the restriction of $f : R \rightarrow \mathbb{P}(V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d)))^\vee = R'$ on R° in this case. Thus f is surjective. Since the Picard numbers of R and R' are two and one respectively, f is a divisorial contraction. Hence $\mathbb{R}_{\geq 0}\alpha$ is an edge of $\text{Mov}(R)$.

To compute the class of the contracted divisor, we compare the canonical divisors on R and R' . Since $r = 1$,

$$K_R = -(n + (2r + 2 - n)d)\alpha - (n - 2r)\beta = -(n + (4 - n)d)\alpha - (n - 2)\beta.$$

On the other hand, $K_{R'} = \mathcal{O}_{R'}(-\dim R' - 1) = \mathcal{O}_{R'}(-nd - n)$ and $f^*\mathcal{O}_{R'}(1) = \alpha$. Hence we have

$$K_R - f^*K_{R'} = (n - 2)(2d\alpha - \beta).$$

Since $n - 2 \geq r = 1$, the class of the contracted divisor is a positive multiple of $2d\alpha - \beta$ and this class spans an edge of $\text{Eff}(R)$. \square

Lemma 5.9. *If $2 \leq r \leq n - 2$, R' is an SQM of R . Furthermore, $\text{Nef}(R') = \mathbb{R}_{\geq 0}\alpha + \mathbb{R}_{\geq 0}(2d\alpha - \beta)$ and $\alpha, 2d\alpha - \beta$ are base point free on R' .*

Proof. By [Sh, Corollary 4.9], the codimension of $R \setminus R^\circ$ in R is r , and that of $R' \setminus R'^\circ$ in R' is s . Hence R' is an SQM of R by $2 \leq r \leq n - 2$. Similar to R , $\text{Nef}(R')$ is spanned by base point free line bundles $\alpha' := c_1(B'_d) - c_1(B'_{d-1})$ and $\beta' := c_1(B'_{d-1})$ for $B'_m = \pi_*(\mathcal{B}'(m))$. Hence it suffices to show that $\alpha' = \alpha$ and $\beta' = 2d\alpha - \beta$ under the identification of $N^1(R)_\mathbb{R}$ and $N^1(R')_\mathbb{R}$.

Similar to f , there exists a morphism $f' : R' \rightarrow \mathbb{P}(\bigwedge^s V^\vee \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)$. By the canonical isomorphism $\mathbb{P}(\bigwedge^r V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee) \xrightarrow{\sim} \mathbb{P}(\bigwedge^s V^\vee \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)$ and the definitions of f, f' , we have a commutative diagram

$$(5.4) \quad \begin{array}{ccc} R & \dashrightarrow & R' \\ f \downarrow & \circlearrowleft & \downarrow f' \\ \mathbb{P}(\bigwedge^r V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee) & \xrightarrow{\sim} & \mathbb{P}(\bigwedge^s V^\vee \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee). \end{array}$$

Hence we have $\alpha = f^*\mathcal{O}(1) = f'^*\mathcal{O}(1) = \alpha'$.

Let $h = c_1(p_1^*\mathcal{O}_{\mathbb{P}^1}(1))$ for $p_1 : \mathbb{P}^1 \times R \rightarrow \mathbb{P}^1$. As in [St] and [Ma], we can write the Chern classes $c_i(\mathcal{A}) = t_i + hu_{i-1}$ for $i = 1, 2$, $t_i \in A^i(R)$, $u_{i-1} \in A^{i-1}(R)$. By [Ma, Lemma 3.1], $t_1 = -\alpha, u_1 = \beta$. In particular, $\beta = \pi_{R*}(c_2(\mathcal{A}))$ holds. Similarly, we have $\beta' = \pi_{R'*}(c_2(\mathcal{A}'))$. Under the natural identifications $A^1(R') = A^1(R'^\circ) = A^1(R^\circ) = A^1(R)$, we have

$$\begin{aligned} \beta' &= \pi_{R'*}(c_2(\mathcal{A}')) = \pi_{R'^\circ*}(c_2(\mathcal{A}'|_{\mathbb{P}^1 \times R'^\circ})) \\ &= \pi_{R^\circ*}(c_2(\mathcal{B}^\vee|_{\mathbb{P}^1 \times R^\circ})) = \pi_{R^\circ*}(c_2(\mathcal{B}|_{\mathbb{P}^1 \times R^\circ})) = \pi_{R*}(c_2(\mathcal{B})) \end{aligned}$$

As written in the proof of [Ma, Lemma 3.1], it holds that $c_2(\mathcal{B}) = t_1^2 - t_2 - h(2dt_1 + u_1)$. Hence we have $\beta' = \pi_{R*}(c_2(\mathcal{B})) = -(2dt_1 + u_1) = 2d\alpha - \beta$. \square

Remark 5.10. To describe $\text{Eff}(R)$, Jow [Jo] used another basis Y, D of $\text{Pic}(R)$, which was introduced by Martínez in [Ma]. The classes Y and D are defined by

$$Y = \pi_{R*}(h \cdot c_1(\mathcal{B})), \quad D = \pi_{R*}(c_2(\mathcal{B})).$$

By Lemma 3.2 in [Ma] and Introduction in [Jo], $Y = \alpha$ and $D = 2d\alpha - \beta$ holds. Hence Y and D are nothing but α' and β' respectively if $2 \leq r \leq n - 2$.

As in Remark 5.2, pr_1, pr_2 in the diagram (5.1) are small contractions for $\lfloor d/s \rfloor + 1 \leq m \leq d - 1$. In the following lemma, we see that f, f' in the diagram (5.4) are small contractions. We note that $f'(R') = f(R) = K_{s,r}^d$ holds by the diagram (5.4) for $2 \leq r \leq n - 2$ under the identification $\mathbb{P}(\bigwedge^r V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee) = \mathbb{P}(\bigwedge^s V^\vee \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)$.

Lemma 5.11. *If $2 \leq r \leq n - 2$, $f : R \rightarrow K_{s,r}^d$ and $f' : R' \rightarrow K_{s,r}^d$ are small contraction.*

Proof. It suffices to show that $f : R \rightarrow K_{s,r}^d$ is a small contraction. By [Sh, Corollary 4.9], the codimension of $R \setminus R^\circ$ in R is $r \geq 2$. Since $K_{s,r}^d$ is normal by Lemma 5.5, it is enough to show that f is an embedding on R° .

Since $\text{Gr}(V, r)$ is embedded into $\mathbb{P}(\bigwedge^r V)$ by the Plücker embedding, $R^\circ = \text{Mor}_d(\mathbb{P}^1, \text{Gr}(V, r))$ is embedded into $\text{Mor}_d(\mathbb{P}^1, \mathbb{P}(\bigwedge^r V))$. Let \bar{R} be the Quot scheme parametrizing all rank 1, degree d quotient sheaves of $(\bigwedge^r V)_{\mathbb{P}^1}$, which is a compactification of $\text{Mor}_d(\mathbb{P}^1, \mathbb{P}(\bigwedge^r V))$ since $\mathbb{P}(\bigwedge^r V) = \text{Gr}(\bigwedge^r V, 1)$. Applying the definition of f to \bar{R} , we have a morphism $\bar{f} : \bar{R} \rightarrow \mathbb{P}(\bigwedge^r V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)$. By the constructions of f, \bar{f} , the diagram

$$\begin{array}{ccc} R^\circ = \text{Mor}_d(\mathbb{P}^1, \text{Gr}(V, r)) & \hookrightarrow & \text{Mor}_d(\mathbb{P}^1, \mathbb{P}(\bigwedge^r V)) \subset \bar{R} \\ & \searrow f|_{R^\circ} & \downarrow \bar{f}|_{\text{Mor}_d(\mathbb{P}^1, \mathbb{P}(\bigwedge^r V))} \\ & & \mathbb{P}(\bigwedge^r V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee). \end{array}$$

is commutative. Applying the proof of Lemma 5.8 to \bar{R} , the restriction $\bar{f}|_{\text{Mor}_d(\mathbb{P}^1, \mathbb{P}(\bigwedge^r V))}$ is an open immersion. Hence $f|_{R^\circ}$ is an embedding and this lemma holds. \square

Lemma 5.12. *Assume $2 \leq r \leq n - 2$. For $\lfloor d/r \rfloor + 1 \leq m' \leq d - 1$, there exists a smooth SQM $R'_{m'}$ of R such that*

$$\text{Nef}(R'_{m'}) = \mathbb{R}_{\geq 0}((d + m')\alpha - \beta) + \mathbb{R}_{\geq 0}((d + m' + 1)\alpha - \beta)$$

and $(d + m')\alpha - \beta$ and $(d + m' + 1)\alpha - \beta$ are base point free on $R'_{m'}$. Furthermore, $\mathbb{R}_{\geq 0}((d + \lfloor d/r \rfloor + 1)\alpha - \beta)$ and $\mathbb{R}_{\geq 0}((d + \lceil d/r \rceil)\alpha - \beta)$ are edges of $\text{Mov}(R)$ and $\text{Eff}(R)$ respectively.

Proof. Applying Proposition 5.1 to R' , we obtain SQMs $R'_{m'}$ of R' such that

$$\text{Nef}(R'_{m'}) = \mathbb{R}_{\geq 0}(-(d - 1 - m')\alpha' + \beta') + \mathbb{R}_{\geq 0}(-(d - m')\alpha' + \beta').$$

Since $\alpha' = \alpha$ and $\beta' = 2d\alpha - \beta$ by the proof of Lemma 5.9, we have the first assertion. The rest part follows by applying Lemma 5.3 to R' . \square

Proof of Theorem 1.5. (1) follows from Lemma 5.1. (2) follows from Lemmas 5.3, 5.6, and 5.8. (3) follows from Lemmas 5.6, and 5.8. (4) follows from Lemma 5.3. (5) follows from Remark 5.2. \square

Proof of Theorem 1.6. (1) follows from Lemmas 5.1, 5.9, 5.12. (2) follows from Lemmas 5.3, 5.12. (3) follows from Lemma 5.3. (4) follows by applying Lemma 5.3 to R' . (5) follows from Remark 5.2. (6) follows by applying Remark 5.2 to R' . (7) follows from Lemma 5.11. \square

Proof of Corollary 1.7. We may assume that $0 \leq r \leq n - 2$ and $d \geq 1$.

By results of Strømme, R satisfies conditions 1) and 2) in the definition of Mori dream spaces. By Theorems 1.5, 1.6, R satisfies condition 3). Hence R is a Mori dream space.

By Theorems 1.5, 1.6 and the description of K_R by α, β , we can check that $-K_R \in \text{Mov}(R) \cap \text{Eff}(R)^\circ$, where $\text{Eff}(R)^\circ$ is the interior of $\text{Eff}(R)$. Hence there exists an SQM R^\dagger of R such that whose anti-canonical divisor is nef and big. Since R^\dagger is smooth, R^\dagger is log Fano. Hence R is log Fano by [Bi, Lemma 2.4]. \square

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